Int. J. Engng Sci. Vol. 19, pp. 1353–1368, 1981
Printed in Great Britain.
0020-7225/81/101353-16$02.00/0

ON THE CONSISTENCY CONDITIONS OF AVERAGING OPERATORS IN 2-PHASE FLOW MODELS AND ON THE FORMULATION OF MAGNETOHYDRODYNAMIC 2-PHASE FLOW

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Abstract—An analysis has been carried out on the space averaged form of 2-phase flow conservation equations in which are also included the electromagnetic fields in the low frequency (MHD) approximation. The analysis considers ensemble and time averages of the space integral operators and attempts to derive their consistency conditions. It is shown that it is possible to rigorously derive the consistency conditions between averaging operators, provided that the stochastic process for the "noise" of the space integral operators can be identified. A plausible choice for this "noise" has been made and the parameters of its description have been identified. The 2-phase flow MHD model is formulated so that it satisfies the consistency conditions of the integral operators and in which the space integral operator is defined over an area. A simplified 1-dimensional MHD model is developed in terms of the conservation equations for the 2-phase mixture. The boundary conditions and constitutive equations which enter into the model are discussed for applications dealing with 2-phase flow in magnetohydrodynamic generators and in magnetically confined plasma fusion reactors.

1. INTRODUCTION

Rigorous formulation of the 2-phase flow conservation equations has for some time been a challenge, and the inclusion of electric and magnetic fields into the theory has not been carried out to date. The difficulty in the formulation is associated with the presence of a large number of interfaces and the apparent random nature of the flow field. For this reason large number of 2-phase flow models have been discussed in the literature in recent years[1–4].

Due to the presence of interfaces in 2-phase flow, in all the proposed models it is recognized that 2-phase flow modeling is conceptually different from the modeling of multicomponent reacting flow mixture of continuum mechanics (Eringen[5]). These models are derived from macroscopic conservation equations of each phase and the interface boundary conditions. The later set of equations specifies the interphase mass, momentum, energy and entropy transfer as well as the boundary conditions for the electromagnetic fields. Integration of these macroscopic equations over space and time segments yields the desired 2-phase flow conservation equations. Thus Ishii[1] carried out time-averaging of the conservation equations and derived both the mixture and 2-fluid 2-phase flow models. Delhaye and Achard[2] integrated the conservation equations over the volume, area and segment, and Chawla and Ishii[4] carried out area-averaging of the ensemble averaged conservation laws. The resulting equations in all these models are expressed in terms of time and space integrals into which the details of the 2-phase flow field have been thrown. Consequently, two new difficulties appear: how to specify constitutive equations for the time and space integrals, and how to select the value of the averaging time interval and the size of the averaging space regions. Different choice of these values will give rise to quantitatively as well as qualitatively different solutions of the differential equations. For these reasons it is important to identify precisely the meanings of integral operators, since only then the results from the theory and the experiment can be compared.

An important initial step has recently been taken by Delhaye and Achard[2] who studied time-averaging and statistical-averaging operators. However, they did not show in which way these two operators can be made to agree with each other. By establishing the conditions under which the averaging operators can be made to agree with each other, it is possible to provide an internal consistency between different 2-phase flow theories and pave the way for their rigorous classification.

This paper has the following objectives:

1. To extend the range of applicability of existing 2-phase flow models by incorporating into the formulation the electric and magnetic fields.

2. To present sufficient conditions for the equivalence of averaging operators in 2-phase flow models.

3. To present a magnetohydrodynamic 2-phase flow model and discuss its constitutive closure conditions for a number of physical applications.

2. FORMULATION OF 2-PHASE FLOW MODELS

The 2-phase mixture will be envisioned as consisting of either the continuous vapor or gas phase in which liquid is dispersed or of the continuous liquid phase in which vapor or gas is dispersed. If this 2-phase mixture consists of liquid metal and gas and if a significant magnetohydrodynamic effect is to be produced from the flow, then the continuous phase must necessarily be the liquid metal, in order to produce continuous-path, high electrical conductivity for the electric current.

A phase will be labeled by the subscript k whose value is 1 or 2 and either value can be associated with the dispersed or continuous phase. The interface between phases will be modeled by a surface of discontinuity. For each phase and for the liquid–vapor interface the macroscopic equations which describe the conservation of mass, momentum, energy and the electromagnetic fields can be written as follows (Eringen[6] and Jackson[7]):

(a) *Conservation of mass, momentum and energy*

For each phase

$$\frac{\partial}{\partial t}(\rho_k \Psi_k) + \nabla \cdot (\rho_k \Psi_k \mathbf{V}_k) + \nabla \cdot J_k - \rho_k \phi_k = B_k \tag{1}$$

where Ψ_k, J_k, ϕ_k and B_k are given in Table 1.

At the interface

$$\sum_{k=1,2} (\dot{m}_k \Psi_k + \hat{n}_k \cdot J_k) = \Delta \tag{2}$$

where

$$\dot{m}_k = \rho_k (\mathbf{V}_k - \mathbf{S}_i) \cdot \hat{n}_k \tag{3}$$

is the interphase mass transfer, $\mathbf{S}_i$ is the interface velocity and H is the mean curvature of the interface. ∇_S is the gradient vector in the surface and Δ is the surface tension term given in Table 1.

(b) *Maxwell's electrodynamic equations*

For each phase

$$\frac{\partial}{\partial t}\Omega_k + \nabla \times \chi_k + \nabla \cdot \Sigma_k = \mu_0 \Lambda_k \tag{4}$$

where Ω_k, χ_k, Σ_k and Λ_k are given in Table 2.

At the liquid-vapor interface

$$\sum_{k=1,2} (\hat{n}_k \times \chi_k + \hat{n}_k \cdot \Sigma_k + \Phi_k \hat{n}_k \cdot \mathbf{S}_i) = \lambda \mu_0 \mathbf{i}_S \tag{5}$$

where $\mathbf{i}_S$ is the interface surface current.

Table 1. Coefficients of the general balance eqn (1) and interface transport eqn (2)

CONSERVATION OF	Ψ_k	J_k	$\rho_k\phi_k$	B_k	Δ
MASS	1	0	0	0	0
LINEAR MOMENTUM	V_k	$-\Pi_k = p_k I - \tau_k$	$\rho_k g + i_k \times B_k$	0	$2H\nu\hat{n}_i + \nabla_S\nu$
ANGULAR MOMENTUM	$\Pi_k = \Pi_k^T$				
ENERGY	$u_k + \frac{V_k^2}{2}$	$q_k - \Pi_k \cdot V_k$	$\rho_k g \cdot V_k + i_k \cdot E_k$	0	$2H\nu\hat{n}_i \cdot S_i + \nabla_S\nu \cdot S_i + \nu\nabla_S \cdot S_i$

Table 2. Coefficients of the Maxwell's electrodynamic eqns (4) and interface boundary conditions (5)

LAW	Ω_k	X_k	Σ_k	Λ_k	Φ_k	λ
FARADAY	B_k	E_k	0	0	$-B_k$	0
AMPERE	0	B_k	0	i_k	$\mu_o\epsilon_o E_k$	1
$\nabla \cdot B_k = 0$	0	0	B_k	0	0	0
$\nabla \cdot E_k = 0$	0	0	E_k	0	0	0

Equations (1)–(5) describe a continuum of the following type: 1. Local field theory. 2. Non-relativistic fluid ($v/c \ll 1$). 3. Low frequency (MHD) approximation (locally neutral plasma, negligible displacement current and negligible ion slip).

(c) *Space, time and ensemble averaged form of conservation equations*

A useful set of conservation equations for the 2-phase flow is obtained by averaging the conservation eqns (1) and (4) over an area which is perpendicular to the main flow direction. Area-averaged form of equations is most useful when analyzing 2-phase flow in internal geometries such as pipes, magnetohydrodynamic channels or in the region between the pin bundles of a nuclear reactor (Chawla and Ishii[4], Dobran[8]).

Area averaging of the conservation eqns (1) and (4) is carried out for each constituent phase over the portion of the total flow area which the phase occupies at time t. Figure 1 illustrates the nomenclature. To interchange the order of integration and differentiation after the area averaging procedure, the mathematical identities of the Appendix A are utilized. Equations (6) and (7) given below show the final product of these operations.

$$\frac{\partial}{\partial t}O\left[\int_{a_k}\rho_k\Psi_k\,\mathrm{d}a\right]+\frac{\partial}{\partial z}O\left[\int_{a_k}\rho_k\Psi_k\mathrm{V}_k\cdot\hat{n}_z\,\mathrm{d}a\right]+\frac{\partial}{\partial z}O\left[\int_{a_k}\cdot J_k\,\mathrm{d}a\right]-O\left[\int_{a_k}\rho_k\phi_k\,\mathrm{d}a\right]$$
$$=O\left[\int_{a_k}B_k\,\mathrm{d}a\right]-O\left[\int_{\xi_i+C_k}(\dot{m}_k\Psi_k+\hat{n}_k\cdot J_k)\frac{\mathrm{d}\xi}{\hat{n}_k\cdot\hat{n}_{k\xi}}\right] \tag{6}$$

$$\frac{\partial}{\partial t}O\left[\int_{a_k}\Omega_k\,\mathrm{d}a\right]+\frac{\partial}{\partial z}O\left[\int_{a_k}\hat{n}_z\times\chi_k\,\mathrm{d}a\right]+\frac{\partial}{\partial z}O\left[\int_{a_k}\hat{n}_z\cdot\Sigma_k\,\mathrm{d}a\right]$$
$$=\mu_0O\left[\int_{a_k}\Lambda_k\,\mathrm{d}a\right]-O\left[\int_{\xi_i+C_k}(\hat{n}_k\times\chi_k+\hat{n}_k\cdot\Sigma_k-\Omega_k\hat{n}_k\cdot\mathrm{S}_i)\frac{\mathrm{d}\xi}{\hat{n}_k\cdot\hat{n}_{k\xi}}\right]. \tag{7}$$

The operator on the quantities inside square brackets in the above equations is just the quantity

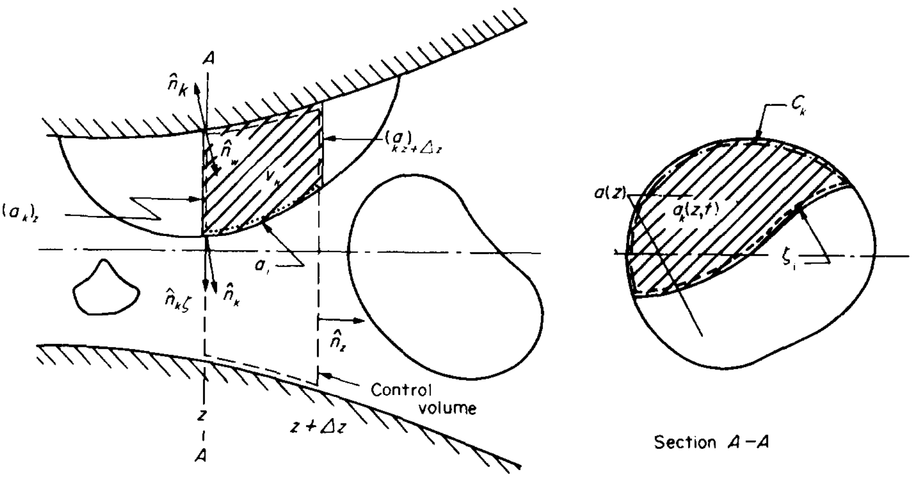

Fig. 1. Variable area channel with 2-phase flow and definition of terms for the area-averaging operation.

inside the brackets, i.e.

$$O\left[\int_{a_k} \theta_k \, da\right] = \int_{a_k} \theta_k \, da. \tag{8a}$$

The form of eqns (6) and (7) is valid if the integrands on the left side of equations in Appendix A are continuous over the region of integration. When the interface becomes tangential to a_k while ξ_i remains finite, the integral does not exist, since no unique value can be assigned to the integrand. These isolated singularities are a consequence of the physical model which assumes that the interface is a surface of discontinuity. If necessary they can be accounted for within the theory. However, due to the turbulent fluctuations of the flow, it is highly improbable that the conditions given above can be met for the occurrence of these singularities.

Time-averaging of eqns (6) and (7) is carried out over an interval of time T. This procedure readily yields a set of equations which is similar to eqns (6) and (7) but where the operator now becomes

$$O\left[\int_{a_k} \theta_k \, da\right] = \frac{1}{T}\int_{t-T/2}^{t+T/2} \int_{a_k} \theta_k \, da \, du. \tag{9a}$$

A third averaging procedure will be introduced shortly. Towards this end it may be noted that in turbulent flow modeling it is considered that the macroscopic variables in the conservation eqns (1) and (4) describe turbulent fields. The same is then true in both the area averaged and time averaged form of eqns (6) and (7). In this way the dependent variables entering into these equations are random variables and can be defined on the parameter space which will be taken to be the time. Equations (6) and (7), therefore, describe stochastic processes; to indicate explicitly which variables are random variables, we will modify the meaning of the operator in eqn (8a) by writing

$$O\left[\int_{a_k} \theta_k \, da\right] = \int_{\bar{\bar{a}}_k} \bar{\bar{\theta}}_k \, da \tag{8b}$$

and the operator in eqn (9a) by writing

$$O\left[\int_{a_k} \theta_k \, da\right] = \frac{1}{T}\int_{t-T/2}^{t+T/2} \int_{\bar{\bar{a}}_k} \bar{\bar{\theta}}_k \, da \, du. \tag{9b}$$

The two bars over $\bar{\bar{a}}_k$ and $\bar{\bar{\theta}}_k$ in (8b) and (9b) stress the fact that both are random variables.

The third operator is now defined on the area-averaged form of conservation equations and operates over the probability space of the random variables. This is equivalent to taking the expectation (ensemble average) of the area averaged equations. If it is assumed that the probability integrals converge in the mean square sense or with the probability one, the result of this operation is that the operator in eqns (6) and (7) is replaced by the following

$$O\left[\int_{a_k} \theta_k \, \mathrm{d}a\right] = E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\}. \tag{10}$$

Recently Chawla and Ishii[4] first carried out ensemble averaging on eqn (1) and then area averaging. Their procedure is obtained by replacing the operator in eqn (6) as follows.

$$O\left[\int_{a_k} \theta_k \, \mathrm{d}a\right] = \int_{a_k} E\{\bar{\bar{\theta}}_k\} \, \mathrm{d}a. \tag{11}$$

In the development of 2-phase flow equations for the flow in pin bundles of a nuclear reactor, Chawla and Ishii assumed that the interfaces are *not* of random nature and, therefore, they felt justified in carrying out ensemble averaging of the conservation equations prior to the area averaging.

3. ANALYSIS OF AREA AVERAGE, TIME AVERAGE AND ENSEMBLE AVERAGE OPERATORS

(a) *Analysis of the ensemble average operator*

In this section a closer examination of the ensemble averaging operator will be carried out. Let us define the stochastic integral by

$$\bar{\bar{F}}_k \equiv \int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a. \tag{12}$$

Assuming that the area integral converges in the mean square sense, then with probability one

$$E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\} = E\left\{\int_{\bar{a}_k} E\{\bar{\bar{\theta}}_k \mid \bar{a}_k\} \, \mathrm{d}a\right\} \tag{13a}$$

where $E\{\bar{\bar{\theta}}_k \mid \bar{a}_k\}$ is the conditional expectation. If the *stochastic correlation coefficient* $\bar{\bar{\eta}}_k$ is defined by the relation

$$\bar{\bar{\eta}}_k \equiv \frac{E\{\bar{\bar{\theta}}_k \mid \bar{a}_k\}}{E\{\bar{\bar{\theta}}_k\}}$$

then eqn (13a) can be written also as

$$E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\} = E\left\{\int_{\bar{a}_k} \bar{\bar{\eta}}_k E\{\bar{\bar{\theta}}_k\} \, \mathrm{d}a\right\}. \tag{13b}$$

If in the above equation we let $\bar{a}_k = a_k$, i.e. the area a_k is an ordinary function and not a random function, then necessarily $\bar{\bar{\eta}}_k = 1$. The expected value in (13b) is now taken over the ordinary function (the value of the ordinary integral) which is just equal to the value of the ordinary function (the integral). This, then, proves assertion (11).

The stochastic correlation coefficient $\bar{\bar{\eta}}_k$ has a value different from unity whenever there is a correlation between the space-time value of fluid variables and $\bar{a}_k$. In the immediate vicinity of the liquid–vapor interface the correlation is strongest and a fluid model might be envisioned with a correlation boundary layer with effective thickness equal to the length over which $\bar{\bar{\eta}}_k$ differs significantly from the value of unity. The structure of turbulent flow determines $\bar{\bar{\eta}}_k$, and the correlation coefficient can be regarded, therefore, as a constitutive quantity. Standard probability arguments can be used to show that if $\bar{\bar{\eta}}_k = 1$, then $\bar{\bar{\theta}}_k$ and $\bar{\bar{a}}_k$ are uncorrelated and not necessarily independent, but if they are independent then it must also follow that they are uncorrelated or $\bar{\bar{\eta}}_k = 1$.

The above exposition clearly demonstrates that care must be exercised in postulating that the ensemble average of the area average quantity is equal to the area average of the ensemble average quantity.

(b) *Relation between the ensemble average and the time average of the area averaged 2-phase flow equations*

In this section the connection between ensemble average and time average operators will be investigated. Specifically, if the time average of the stochastic process (12) is defined by

$$\bar{\bar{F}}_{kT} \equiv \frac{1}{T} \int_{t-T/2}^{t+T/2} \bar{\bar{F}}_k \, du \tag{14}$$

then it is of interest to be able to seek conditions under which eqns (6) and (7) are identical no matter whether the time average operator (9b) or the ensemble average operator (10) is utilized. Hence we wish to investigate the requirements which will give

$$E\{\bar{\bar{F}}_k\} = \bar{\bar{F}}_{kT} \tag{15a}$$

$$\frac{\partial}{\partial t} E\{\bar{\bar{F}}_k\} = \frac{\partial}{\partial t} \bar{\bar{F}}_{kT}. \tag{15b}$$

The proof of the above propositions is facilitated by decomposing the stochastic process (12) into an ordinary function of space and time and a stochastic "noise" function which is constructed as follows

$$\bar{\bar{F}}_k \equiv f_k + \bar{\bar{n}}_k \tag{16}$$

$$E\{\bar{\bar{F}}_k\} = f_k, \quad E\{\bar{\bar{n}}_k\} = 0. \tag{17a}$$

Since in this paper we will deal only with differentiable (in the ordinary sense) stochastic processes, it is permissible also to write using above equations.

$$E\left\{\frac{\partial \bar{\bar{F}}_k}{\partial t}\right\} = \frac{\partial}{\partial t} E\{\bar{\bar{F}}_k\} = \frac{\partial f_k}{\partial t} \tag{17b}$$

$$E\{\bar{\bar{F}}_{kT}\} = \frac{1}{T} \int_{t-T/2}^{t+T/2} f_k \, du \tag{18a}$$

$$\frac{\partial}{\partial t} E\{\bar{\bar{F}}_{kT}\} = \frac{\partial}{\partial t} \frac{1}{T} \int_{t-T/2}^{t+T/2} f_k \, du. \tag{18b}$$

Combining (14), (16) and (17a); and (14), (16) and (18b) yields, respectively

$$\bar{\bar{F}}_{kT} = E\{\bar{\bar{F}}_{kT}\} + \frac{1}{T} \int_{t-T/2}^{t+T/2} \bar{\bar{n}}_k \, du \tag{19a}$$

$$\frac{\partial \bar{\bar{F}}_{kT}}{\partial t} = \frac{\partial}{\partial t} E\{\bar{\bar{F}}_{kT}\} + \frac{1}{T} \int_{t-T/2}^{t+T/2} \frac{\partial \bar{\bar{n}}_k}{\partial u} \, du. \tag{19b}$$

Proposition. With probability P_1, $\bar{\bar{F}}_{kT}$ can be approximated by $E\{\bar{\bar{F}}_{kT}\}$, and with probability P_2 $(\partial/\partial t)\bar{\bar{F}}_{kT}$ can be approximated by $(\partial/\partial t)E\{\bar{\bar{F}}_{kT}\}$.

Proof. Indeed, using Chebyshev inequality [9]

$$P_1\{|\bar{\bar{F}}_{kT} - E\{\bar{\bar{F}}_{kT}\}| < \epsilon_1\} \geq 1 - \frac{\sigma^2_{\bar{F}_{kT}}}{\epsilon_1^2} \tag{19c}$$

$$P_2\left\{\left|\frac{\partial \bar{\bar{F}}_{kT}}{\partial t} - \frac{\partial}{\partial t} E\{\bar{\bar{F}}_{kT}\}\right| < \epsilon_2\right\} \geq 1 - \frac{\sigma^2_{\partial \bar{F}_{kT}/\partial t}}{\epsilon_2^2} \tag{19d}$$

with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ completes the proof.

$\sigma_{\bar{F}_{kT}}$ and $\sigma_{\partial \bar{F}_{kT}/\partial t}$ are the variances of the stochastic processes (19a) and (19b), respectively. Since these variances are only functions of the noise spectra of the processes (19a) and (19b), their estimation is not difficult.

The *noise*, $\bar{n}_k$, fluctuates generally rapidly with time. Because by construction the ensemble mean of $\bar{n}_k$ is zero, it can physically be argued that the product $\bar{n}_k(t)\bar{n}_k(t+\tau)$ is as much positive as it is negative and generally independent of time t, that is, $\bar{n}_k$ is a stationary process. Also $E\{\bar{n}_k(t)\bar{n}_k(t+\tau)\}$ should be greatest for small values of τ and smallest for large values of τ. Certainly, if there were any periodic frequency component of the process in $\bar{n}_k$, then whenever the period of this component is equal to the integer value of τ, the autocorrelation function would exhibit a peak and the decomposition (16) would *not* be acceptable, since we would require that the periodic component of the process be included in f_k and not in $\bar{n}_k$.

There are infinite numbers of distribution functions for the autocorrelation function $R_{\bar{n}_k}(\tau) = E\{\bar{n}_k(t)\bar{n}_k(t+\tau)\}$ which satisfy the above requirements. In Appendix B an analysis is presented by postulating that $R_{\bar{n}_k}$ is normally distributed with zero mean and variance $\sigma_{\bar{n}_k}$. It is shown there that it is possible to select the variances of processes (19a, b) to be arbitrarily small by a suitable choice of $\sigma_{\bar{n}_k}/T$ and $R_{\bar{n}_k}(0)/T^2$ where $R_{\bar{n}_k}(0)$ is the average power of the noise $\bar{n}_k$.

To complete the proof which leads to identities (15a, b) the ordinary function, f_k, of the process (16) need to be examined.

For this purpose, and without loss of generality, let the decomposition of f_k be harmonic

$$f_k = A_k(x)\sin(w_k t)$$

with the characteristic frequency of the process $w_k/2\pi$. The *gain factors* defined by the relations

$$G_1 \equiv \left|\frac{\dfrac{1}{T}\displaystyle\int_{t-T/2}^{t+T/2} f_k \, du}{f_k}\right| = \frac{\left|\sin\left(\dfrac{w_k T}{2}\right)\right|}{\dfrac{w_k T}{2}} \tag{21a}$$

$$G_2 \equiv \left|\frac{\dfrac{1}{T}\displaystyle\int_{t-T/2}^{t+T/2} \dfrac{\partial f_k}{\partial u} \, du}{\dfrac{\partial f_k}{\partial t}}\right| = \frac{\left|\sin\left(\dfrac{w_k T}{2}\right)\right|}{\dfrac{w_k T}{2}} \tag{21b}$$

have identical values. We will require that the averaging time period T is selected in such a way as to produce a gain factor close to one, since only then can the important information of the process in our solution to the problem be retained. Thus, it is necessary that

$$\frac{\left|\sin\left(\dfrac{w_k T}{2}\right)\right|}{\dfrac{w_k T}{2}} > G_{\min} \approx 1. \tag{21c}$$

The assertions (15) are now proved by the following sequence of steps

$$E\{\bar{F}_k\} \overset{(17a)}{=} f_k \overset{\substack{(21a)\\(21c)}}{=} \frac{1}{T}\int_{t-T/2}^{t+T/2} f_k \, du \overset{(18a)}{=} E\{\bar{\bar{F}}_{kT}\} \overset{(19c)}{=} \bar{\bar{F}}_{kT} \tag{15a}$$

$$\frac{\partial E\{\bar{\bar{F}}_k\}}{\partial t} \overset{(17b)}{=} \frac{\partial f_k}{\partial t} \overset{(21b)}{\underset{(21c)}{=}} \frac{1}{T}\int_{t-T/2}^{t+T/2} \frac{\partial f_k}{\partial u}\, du \overset{(18b)}{=} \frac{\partial}{\partial t} E\{\bar{\bar{F}}_{kT}\}$$

$$\overset{(19d)}{=} \frac{\partial \bar{\bar{F}}_{kT}}{\partial t}. \tag{15b}$$

The results (15a, b) are very important since they allow a self-consistent formulation of the 2-phase flow theories.

(c) *Discussion of the consequences of eqns* (15a, b)

If conditions which lead to results (15a, b) are satisfied, the implication of these equations is that the ensemble mean of a stochastic integral can be obtained by computing the area integral over only a *typical* sample function of stochastic processes $\bar{a}_k$ and $\bar{\bar{\theta}}_k$. In mathematical form, if δ signifies a *typical* sample function of the stochastic process, eqns (15a, b) can be written as follows

$$E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\} = \frac{1}{T}\int_{t-T/2}^{t+T/2} \int_{a_k(\delta,u)} \theta_k(\delta, u)\, \mathrm{d}a\, \mathrm{d}u = \int_{a_k(\delta,t)} \theta_k(\delta, t)\, \mathrm{d}a \tag{15c}$$

$$\frac{\partial}{\partial t} E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\} = \frac{\partial}{\partial t}\int_{a_k(\delta,t)} \theta_k(\delta, t). \tag{15d}$$

These equations place a restriction on the selection of a *typical* sample function over which the time-averaging is permissible. Experimentally this can be accomplished by carrying out time-averaging over various subintervals of T and comparing the results with the average over the entire record T. If these averages are the same, the selected sample function is typical. In fast transients of the system, the above procedure will not be possible and in this case an extrapolation of the theory is required.

If the stochastic processes $\bar{a}_k$ and $\bar{\bar{\theta}}_k$ are decomposed similarly to the eqn (16) and if they satisfy the same probability requirements (19c, d) and the gain factor requirement (21c), then from the set of equations which are similar to eqns (15a, b) it can be concluded that eqns (15c, d) can also be written as follows

$$E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\} = \int_{E\{\bar{a}_k\}} E\{\bar{\bar{\theta}}_k\}\, \mathrm{d}a \tag{15e}$$

$$\frac{\partial}{\partial t} E\left\{\int_{\bar{a}_k} \bar{\bar{\theta}}_k \, \mathrm{d}a\right\} = \frac{\partial}{\partial t}\int_{E\{\bar{a}_k\}} E\{\bar{\bar{\theta}}_k\}\, \mathrm{d}a. \tag{15f}$$

These results are slightly different from the model proposed by Chawla and Ishii[4] in that the area integral over which the ensemble mean of a physical quantity is to be evaluated should correspond to the "average" area. In the theory of the above authors no mention is made of how to select a_k. The theory presented herein leads to the natural selection.

It is clear from the preceding exposition that the replacement of area integrals by segment or volume integrals will not introduce any new conceptually different results. The area integrals can be replaced by the generalized (Lebesque) measure.

4. A MODEL FOR THE MAGNETOHYDRODYNAMIC 2-PHASE FLOW

In the last section of this paper a magnetohydrodynamic 2-phase flow model is presented. The model is assumed to satisfy the consistency conditions which lead to eqns (15e, f). The presentation of a 2-phase flow MHD model is timely in view of the recent interest in liquid metal–gas flow in MHD generators (Dobran[8], Petrick *et al.*[10], Pierson *et al.*[11] and Dunn[12]), and the use of liquid metal as a primary blanket coolant in a magnetically confined plasma fusion reactor (Chan[13] and Chao[14]).

Assumptions

1. Flow is predominantly in one direction only, z-direction in Fig. 1.
2. The density and pressure have flat profiles along any cross-sectional plane perpendicular to the flow and they are uncorrelated with any other flow variable: $E\{\bar{\rho}\bar{\theta}\} = E\{\bar{\rho}\}E\{\bar{\theta}\}$, $E\{\bar{p}\bar{\theta}\} = E\{\bar{p}\}E\{\bar{\theta}\}$.
3. Negligible surface tension effects.
4. The boundary of the duct is stationary and impermeable to mass.
5. Electric and magnetic fields are uncorrelated with the electric current and fluid velocity, and have flat profiles.
6. Ohm's law for each phase

$$\mathbf{i}_k = \sigma_k(\mathbf{E}_k + \mathbf{V}_k \times \mathbf{B}_k) - \mu_k \mathbf{i}_k \times \mathbf{B}_k$$

has a negligible contribution from the Hall effect (last term in the above equation).

Let us define

$$\int_{E\{\bar{a}_k\}} E\{\bar{\bar{\theta}}_k\}\,\mathrm{d}a \equiv E\{\bar{a}_k\}\langle E\{\bar{\bar{\theta}}_k\}\rangle = a\tilde{\alpha}_k\langle\tilde{\theta}_k\rangle \tag{22}$$

where a is the cross-sectional area of the duct and $\tilde{\alpha}_k$ is the void fraction defined by

$$\tilde{\alpha}_k \equiv \frac{E\{\bar{a}_k\}}{a}.$$

With the above definitions, eqns (6) and (7) become

$$\begin{aligned}&\frac{\partial}{\partial t}(a\tilde{\alpha}_k\tilde{\rho}_k\langle\tilde{\Psi}_k\rangle) + \frac{\partial}{\partial z}(a\tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{\Psi_k\mathbf{V}_k}\rangle\cdot\hat{n}_z) + \frac{\partial}{\partial z}(a\tilde{\alpha}_k\hat{n}_z\cdot\langle\tilde{J}_k\rangle) - a\tilde{\alpha}_k\tilde{\rho}_k\langle\tilde{\phi}_k\rangle \\ &= a\tilde{\alpha}_k\langle\tilde{B}_k\rangle - \int_{E\{\overline{\overline{\xi_i + C_k}}\}} E\left\{\left(\overline{\overline{\frac{\dot{m}_k\Psi_k + \hat{n}_k\cdot J_k}{\hat{n}_k\ \ \hat{n}_{k\xi}}}}\right)\right\}\mathrm{d}\xi\end{aligned} \tag{23}$$

$$\begin{aligned}&\frac{\partial}{\partial t}(a\tilde{\alpha}_k\langle\tilde{\Omega}_k\rangle) + \frac{\partial}{\partial z}(a\tilde{\alpha}_k\hat{n}_z\times\langle\tilde{\chi}_k\rangle + a\tilde{\alpha}_k\hat{n}_z\cdot\langle\tilde{\Sigma}_k\rangle) \\ &= \mu_0 a\tilde{\alpha}_k\langle\tilde{\Lambda}_k\rangle - \int_{E\{\overline{\overline{\xi_i + C_k}}\}} E\left\{\left(\overline{\overline{\frac{\hat{n}_k\times\chi_k + \hat{n}_k\cdot\Sigma_k - \Omega_k\hat{n}_k\cdot\mathbf{S}_i}{\hat{n}_k\cdot\hat{n}_{k\xi}}}}\right)\right\}\mathrm{d}\xi.\end{aligned} \tag{24}$$

The governing conservation equations for the 2-phase *mixture* are obtained by summing up the constituent phases in eqns (23) and (24). In this process the integrals which contain ξ_i disappear as a consequence of the interface transport conditions (2) and (5) and assumption 3. The integrals over the boundary of the duct C_k which contain $\dot{m}_k$ also vanish because of assumption 4. Hence

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(a\sum_{k=1,2}\tilde{\alpha}_k\tilde{\rho}_k\langle\tilde{\Psi}_k\rangle\Big) + \frac{\partial}{\partial z}\Big(a\sum_{k=1,2}\tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{\Psi_k\mathbf{V}_k}\rangle\cdot\hat{n}_z\Big) + \frac{\partial}{\partial z}\Big(a\sum_{k=1,2}\tilde{\alpha}_k\hat{n}_z\cdot\langle\tilde{J}_k\rangle\Big) - a\sum_{k=1,2}\tilde{\alpha}_k\tilde{\rho}_k\langle\tilde{\phi}_k\rangle \\ &= a\sum_{k=1,2}\tilde{\alpha}_k\langle\tilde{B}_k\rangle - \sum_{k=1,2}\int_{E\{\bar{C}_k\}} E\left\{\left(\overline{\overline{\frac{\hat{n}_k\cdot J_k}{\hat{n}_k\cdot\hat{n}_{k\xi}}}}\right)\right\}\mathrm{d}\xi\end{aligned} \tag{25}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(a\sum_{k=1,2}\tilde{\alpha}_k\langle\tilde{\Omega}_k\rangle\Big) + \frac{\partial}{\partial z}\Big(a\sum_{k=1,2}\tilde{\alpha}_k\hat{n}_z\times\langle\tilde{\chi}_k\rangle + a\sum_{k=1,2}\tilde{\alpha}_k\hat{n}_z\cdot\langle\tilde{\Sigma}_k\rangle\Big) \\ &= \mu_0 a\sum_{k=1,2}\tilde{\alpha}_k\langle\tilde{\Lambda}_k\rangle - \sum_{k=1,2}\int_{E\{\bar{C}_k\}} E\left\{\left(\overline{\overline{\frac{\hat{n}_k\times\chi_k + \hat{n}_k\cdot\Sigma_k}{\hat{n}_k\cdot\hat{n}_{k\xi}}}}\right)\right\}\mathrm{d}\xi - \lambda\int_{E\{\bar{\xi}_i\}} E\left\{\left(\overline{\overline{\frac{\mu_0\mathbf{i}_S}{\hat{n}_1\cdot\hat{n}_{1\xi}}}}\right)\right\}\mathrm{d}\xi.\end{aligned} \tag{26}$$

The set of eqns (23) and (24) form the basis for the study of a *2-fluid model* formulation of MHD. This study will be presented in the future. Here, instead, eqns (25) and (26) will be reduced to what is commonly known in the literature as the *drift flux model*. In the absence of electromagnetic fields, this model has been found to predict the 2-phase flow behavior accurately when there is a strong thermal–hydraulic coupling between the phases.

The drift flux model is formulated in terms of the mixture conservation of mass, mixture momentum equation, mixture energy equation, the continuity equation of one of the phases and the mixture form of Maxwell's equations. The reduction process which leads to the final set of equations involves long algebraic manipulations, and only highlights will be given here.

(a) *Kinematics*

The mixture mass density and velocity are defined by comparing the 2-phase mixture continuity eqn (32) with the continuity equation for the single phase flow.

$$\tilde{\rho}_m \equiv \sum_{k=1,2} \tilde{\alpha}_k \tilde{\rho}_k \tag{27}$$

$$\tilde{\mathbf{V}}_m \equiv \frac{1}{\tilde{\rho}_m} \sum_{k=1,2} \tilde{\alpha}_k \tilde{\rho}_k \langle \tilde{\mathbf{V}}_k \rangle. \tag{28}$$

Velocity of the center of volume is defined as

$$\langle \tilde{\mathbf{J}} \rangle \equiv \sum_{k=1,2} \tilde{\alpha}_k \langle \tilde{\mathbf{V}}_k \rangle \tag{29}$$

and the *dispersed phase drift velocity* is defined by

$$\tilde{\mathbf{V}}_{dj} \equiv \langle \tilde{\mathbf{V}}_d \rangle - \langle \tilde{\mathbf{J}} \rangle. \tag{30}$$

Combining eqns (28)–(30) the dispersed phase and continuous phase velocities are expressed by the relations

$$\langle \tilde{\mathbf{V}}_d \rangle = \tilde{\mathbf{V}}_m + \frac{\tilde{\rho}_c}{\tilde{\rho}_m} \tilde{\mathbf{V}}_{dj}, \quad \langle \tilde{\mathbf{V}}_c \rangle = \tilde{\mathbf{V}}_m - \frac{\tilde{\rho}_d \tilde{\alpha}_d}{\tilde{\rho}_m (1 - \tilde{\alpha}_d)} \tilde{\mathbf{V}}_{dj}. \tag{31a, b}$$

The model will be formulated in terms of the mean velocity and the drift velocity. The later velocity field is convenient since its constitutive expression is usually very simple [15].

(b) *Conservation of mass equations*

Utilizing the coefficients for the conservation of mass from Table 1 in eqn (25) results in an equation for the conservation of mass for the mixture

$$\frac{\partial}{\partial t}(a \tilde{\rho}_m) + \frac{\partial}{\partial z}(a \tilde{\rho}_m \tilde{w}_m) = 0 \tag{32}$$

where $\tilde{w}_m = \tilde{\mathbf{V}}_m \cdot \hat{n}_z$. The continuity equation for the dispersed phase follows from eqn (23), Table 1 and eqns (31), i.e.

$$\frac{\partial}{\partial t}(a \tilde{\alpha}_d \tilde{\rho}_d) + \frac{\partial}{\partial z}(a \tilde{\alpha}_d \tilde{\rho}_d \tilde{w}_m) = a \tilde{\Gamma}_d - \frac{\partial}{\partial z}\left(a \frac{\tilde{\alpha}_d \tilde{\rho}_d \tilde{\rho}_c}{\tilde{\rho}_m} \tilde{w}_{dj}\right) \tag{33}$$

where $\tilde{w}_{dj} = \tilde{\mathbf{V}}_{dj} \cdot \hat{n}_z$ and $\tilde{\Gamma}_d$ is the dispersed phase mass generation rate per unit volume defined by

$$a \tilde{\Gamma}_d \equiv -\int_{E\{\tilde{\xi}_i\}} E\left\{\overline{\overline{\left(\frac{\dot{m}_k}{\hat{n}_k \cdot \hat{n}_{k\xi}}\right)}}\right\} \mathrm{d}\xi.$$

(c) *Mixutre momentum equation*

The momentum equation for the mixture is obtained from eqn (25) and Table 1. The result is

$$\frac{\partial}{\partial t}(a\tilde{\rho}_m\tilde{w}_m)+\frac{\partial}{\partial z}(a\tilde{\rho}_m\tilde{w}_m^2)=-a\frac{\partial\tilde{p}}{\partial z}+\frac{\partial}{\partial z}(a\tilde{\tau}_{mzz})+a\tilde{\rho}_m g_z-\frac{\partial}{\partial z}\left(a\frac{\tilde{\rho}_d\tilde{\rho}_c\tilde{\alpha}_d}{\tilde{\rho}_m(1-\tilde{\alpha}_d)}\tilde{w}_{dj}^2\right)$$
$$+a(\tilde{\mathbf{i}}_m\times\tilde{\mathbf{B}})_z+\widetilde{\tau_w\xi_w}-\frac{\partial}{\partial z}\left(a\sum_{k=1,2}\operatorname{cov}(\alpha_k\tilde{\rho}_k\tilde{w}_k\tilde{w}_k)\right). \tag{34}$$

In the above equation the *mean axial shear stress* is defined by

$$\tilde{\tau}_{mzz}\equiv\sum_{k=1,2}\tilde{\alpha}_k\langle\tilde{\tau}_{kzz}\rangle.$$

The *wall shear stress* is defined by

$$\widetilde{\tau_w\xi_w}\equiv\sum_{k=1,2}\int_{E\{\bar{C}_k\}}E\left\{\overline{\overline{\left(\frac{\hat{n}_k\cdot\boldsymbol{\pi}_k}{\hat{n}_k\cdot\hat{n}_{k\xi}}\right)}}\right\}\cdot\hat{n}_z\,\mathrm{d}\xi$$

and the *mean electric current* is given by the expression

$$\tilde{\mathbf{i}}_m\equiv\sum_{k=1,2}\tilde{\alpha}_k\langle\tilde{\mathbf{i}}_k\rangle.$$

The covariance term

$$\operatorname{cov}(\tilde{\alpha}_k\tilde{\rho}_k\tilde{w}_k\tilde{w}_k)\equiv\tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{w_kw_k}\rangle-\tilde{\alpha}_k\tilde{\rho}_k\langle\tilde{w}_k\rangle\langle\tilde{w}_k\rangle$$

represents the effect of nonuniformity in the velocity profile across the duct cross-sectional area, $\langle\tilde{w}_k\tilde{w}_k\rangle-\langle\tilde{w}_k\rangle\langle\tilde{w}_k\rangle$, and the effect of turbulent momentum transport, $\langle\widetilde{w_k'w_k'}\rangle$.

(d) *Mixutre enthalpy equation*

The steps which lead to the enthalpy equation for the 2-phase mixture are as follows:

1. Form the *mixture internal energy equation* from eqn (25) and Table 1.
2. Form the *mixture kinetic energy equation* by first forming the kinetic energy equation for each phase, area-average and ensemble-average the kinetic energy equation for each phase, and then add the averaged kinetic energy equations.
3. Subtract the mixture kinetic energy equation from the mixture internal energy equation.

The result is

$$\frac{\partial}{\partial t}(a\tilde{\rho}_m\tilde{h}_m)+\frac{\partial}{\partial z}(a\tilde{\rho}_m\tilde{h}_m\tilde{w}_m)=-\frac{\partial}{\partial z}(a\tilde{q}_{mz})+\widetilde{q_w\xi_w}-\frac{\partial}{\partial z}\left(a\frac{\tilde{\alpha}_d\tilde{\rho}_d\tilde{\rho}_c}{\tilde{\rho}_m}\Delta\tilde{h}_{dc}\tilde{w}_{dj}\right)+a\frac{\partial\tilde{p}}{\partial t}$$
$$+a\left(\tilde{w}_m+\frac{\tilde{\alpha}_d(\tilde{\rho}_c-\tilde{\rho}_d)}{\tilde{\rho}_m}\tilde{w}_{dj}\right)\frac{\partial\tilde{p}}{\partial z}+a\tilde{\Phi}^v+a\tilde{\Phi}^i$$
$$-\frac{\partial}{\partial z}\left(a\sum_{k=1,2}\operatorname{cov}(\tilde{\alpha}_k\tilde{\rho}_k\tilde{h}_k\tilde{w}_k)\right)+a\sum_{k=1,2}\tilde{\alpha}_k\left\langle\widetilde{\left(\frac{i_k^2}{\sigma_k}\right)}\right\rangle. \tag{35}$$

The viscous dissipation function, the interfacial mechanical energy transfer function and heat fluxes are defined, respectively, by

$$\tilde{\Phi}^v\equiv\sum_{k=1,2}\tilde{\alpha}_k\langle\widetilde{\boldsymbol{\pi}_k:\nabla\mathbf{V}_k}\rangle$$
$$a\tilde{\Phi}^i\equiv\int_{E\{\bar{\xi}_i\}}\sum_{k=1,2}E\left\{\overline{\overline{\left(\frac{(V_k^2/2)\dot{m}_k-\hat{n}_k\cdot(\boldsymbol{\pi}_k\cdot\mathbf{V}_k)}{\hat{n}_k\cdot\hat{n}_{k\xi}}\right)}}\right\}\mathrm{d}\xi$$

$$\tilde{q}_{mz} \equiv \hat{n}_z \cdot \sum_{k=1,2} \tilde{\alpha}_k \langle \tilde{\mathbf{q}}_k \rangle$$

$$\widetilde{q_w \xi_w} \equiv - \sum_{k=1,2} \int_{E\{\check{C}_k\}} E\left\{ \overline{\overline{\left(\frac{\hat{n}_k \cdot \mathbf{q}_k}{\hat{n}_k \cdot \hat{n}_{k\xi}} \right)}} \right\} \mathrm{d}\xi.$$

The *mixture enthalpy* is defined by

$$\tilde{h}_m \equiv \frac{\sum_{k=1,2} \tilde{\alpha}_k \tilde{\rho}_k \langle \tilde{h}_k \rangle}{\tilde{\rho}_m} \tag{36}$$

and the enthalpy difference is given by

$$\Delta \tilde{h}_{dc} \equiv \langle \tilde{h}_d \rangle - \langle \tilde{h}_c \rangle.$$

The covariance term is defined by

$$\mathrm{cov}(\tilde{\alpha}_k \tilde{\rho}_k \tilde{h}_k \tilde{w}_k) = \tilde{\alpha}_k \tilde{\rho}_k \langle \widetilde{h_k w_k} \rangle - \tilde{\alpha}_k \tilde{\rho}_k \langle \tilde{h}_k \rangle \langle \tilde{w}_k \rangle$$

and is responsible for the turbulent energy transport and the nonuniformity in temperature and flow fields across the duct cross-sectional area.

(e) *Maxwell's electrodynamic equations and Ohm's law*

From Table 2 and eqn (26) the Maxwell's equations for the 2-phase mixture become

$$\frac{\partial}{\partial t}(a\tilde{\mathbf{B}}) + \frac{\partial}{\partial z}(a\hat{n}_z \times \tilde{\mathbf{E}}) = - \sum_{k=1,2} \int_{E\{\check{C}_k\}} E\left\{ \overline{\overline{\left(\frac{\hat{n}_k \times \mathbf{E}_k}{\hat{n}_k \cdot \hat{n}_{k\xi}} \right)}} \right\} \mathrm{d}\xi \tag{37}$$

$$\frac{\partial}{\partial z}(a\hat{n}_z \times \tilde{\mathbf{B}}) = \mu_0 a \tilde{\mathbf{i}}_m - \sum_{k=1,2} \int_{E\{\check{C}_k\}} E\left\{ \overline{\overline{\left(\frac{\hat{n}_k \times \mathbf{B}_k}{\hat{n}_k \cdot \hat{n}_{k\xi}} \right)}} \right\} \mathrm{d}\xi - \int_{E\{\bar{\xi}_i\}} \mu_0 E\left\{ \overline{\overline{\left(\frac{\mathbf{i}_S}{\hat{n}_1 \cdot \hat{n}_{1\xi}} \right)}} \right\} \mathrm{d}\xi \tag{38}$$

$$\frac{\partial}{\partial z}(a\hat{n}_z \cdot \tilde{\mathbf{B}}) = - \sum_{k=1,2} \int_{E\{\check{C}_k\}} E\left\{ \overline{\overline{\left(\frac{\hat{n}_k \cdot \mathbf{B}_k}{\hat{n}_k \cdot \hat{n}_{k\xi}} \right)}} \right\} \mathrm{d}\xi \tag{39}$$

$$\frac{\partial}{\partial z}(a\hat{n}_z \cdot \tilde{\mathbf{E}}) = - \sum_{k=1,2} \int_{E\{\check{C}_k\}} E\left\{ \overline{\overline{\left(\frac{\hat{n}_k \cdot \mathbf{E}_k}{\hat{n}_k \cdot \hat{n}_{k\xi}} \right)}} \right\} \mathrm{d}\xi. \tag{40}$$

Excluding the Hall effect, Ohm's law for the mixture is

$$\begin{aligned} \tilde{\mathbf{i}}_m &= \tilde{\sigma}_m \tilde{\mathbf{E}} + (\tilde{\alpha}_d \langle \tilde{\sigma}_d \rangle \tilde{k}_d + (1 - \tilde{\alpha}_d)\langle \tilde{\sigma}_c \rangle \tilde{k}_c) \tilde{w}_m \hat{n}_z \times \tilde{\mathbf{B}} \\ &\quad + \frac{\tilde{\alpha}_d \tilde{w}_{dj}}{\tilde{\rho}_m} (\tilde{\rho}_c \langle \tilde{\sigma}_d \rangle \tilde{k}_d - \tilde{\rho}_d \langle \tilde{\sigma}_c \rangle \tilde{k}_c) \hat{n}_z \times \tilde{\mathbf{B}}. \end{aligned} \tag{41}$$

$\tilde{k}_d$ and $\tilde{k}_c$ are the *electrical conductivity-flow correlation coefficients* defined by Dobran[8].

$$\tilde{k}_d \equiv \frac{\langle \widetilde{\sigma_d w_d} \rangle}{\langle \tilde{\sigma}_d \rangle \langle \tilde{w}_d \rangle}, \quad \tilde{k}_c \equiv \frac{\langle \widetilde{\sigma_c w_c} \rangle}{\langle \tilde{\sigma}_c \rangle \langle \tilde{w}_c \rangle}.$$

The mean electrical conductivity is given as

$$\tilde{\sigma}_m \equiv \sum_{k=1,2} \tilde{\alpha}_k \langle \tilde{\sigma}_k \rangle.$$

(f) *Discussion*

Some further simplification of the equations presented in Sections 4(c, d) is possible if it is assumed that the velocity has a flat profile. In that case $\langle\tilde{w}_k\tilde{w}_k\rangle = \langle\tilde{w}_k\rangle\langle\tilde{w}_k\rangle$ and in the momentum equation, eqn (34), the cov term becomes

$$\sum_{k=1,2} \operatorname{cov}(\tilde{\alpha}_k\tilde{\rho}_k\tilde{w}_k\tilde{w}_k) = \sum_{k=1,2} \tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{w_k'w_k'}\rangle$$

since the velocity can be decomposed into the mean and a fluctuating component (noise), $w_k = \tilde{w}_k + w_k'$. $\Sigma_{k=1,2}\,\tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{w_k'w_k'}\rangle$ represents the area mean of the velocity correlation and is responsible for the turbulent momentum transport. Also, in the energy equation, eqn (35), the cov term becomes

$$\sum_{k=1,2} \operatorname{cov}(\tilde{\alpha}_k\tilde{\rho}_k\tilde{w}_k\tilde{h}_k) = \sum_{k=1,2} \tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{w_k'h_k'}\rangle$$

and represents the effect of the turbulent energy transport. Velocity and energy correlation terms are functions of the local turbulent structure of the 2-phase flow and enter into the model as constitutive quantities. Future work is needed to identify the values of these terms. A good procedure might be to classify their values according to the 2-phase flow regime.

The integrals over the duct boundary perimeter C_k (Fig. 1) in the conservation eqns (23) and (24) are evaluated by specifying the boundary conditions. In the model the shear stress at the wall and the wall heat flux must be specified *a priori* through the constitutive relations. The constitutive equations should, as usual, satisfy the axioms laid down, as for example, by Eringen[6].

The two continuity equations in the formulation of the drift flux model are required because of the presence of two velocity fields. In the model the dispersed phase drift velocity can be specified as a constitutive quantity (see Ishii[15]) and one of the continuity equations can then be used to solve for the void fraction $\tilde{\alpha}_d$. In the case of reacting flow $\tilde{\Gamma}_d$ needs to be specified also, unless the flow is in thermal equilibrium. In the latter case, $\tilde{\Gamma}_d$ is determined from the enthalpy eqn (35).

The axial shear stress term $\tilde{\tau}_{mzz}$ in eqn (34) is of lesser significance than the axial turbulent shear stress term $\Sigma_{k=1,2}\,\tilde{\alpha}_k\tilde{\rho}_k\langle\widetilde{w_k'w_k'}\rangle$ which arises from the $\Sigma_{k=1,2}\operatorname{cov}(\tilde{\alpha}_k\tilde{\rho}_k\tilde{w}_k\tilde{w}_k)$ expression. Similarly, in eqn (35) the axial heat conduction term $\tilde{g}_{mzz}$ is of lesser importance than the axial turbulence energy transport term $\Sigma_{k=1,2}\,\tilde{\alpha}_k\tilde{\rho}_k\langle h_k'w_k'\rangle$. These deductions are plausible in view of the highly irregular nature of the 2-phase mixture.

In the electrodynamic eqns (37)–(40) the electric and magnetic fields at the wall of the duct need to be specified also in order to evaluate the integrals over C_k. These fields are determined from the electrical and magnetic state of the external electrical circuit of the duct. It is important to realize this point, since the external electrical circuit configuration of the duct will have a profound effect on the flow behavior.

The surface current in the last term on the r.h.s. of the eqn (38) is insignificant unless the interface electrical conductivity is much larger than the conductivity of *both* phases.

In applications involving 2-phase flow in magnetohydrodynamic generators ((Refs. [8, 10]), the continuous phase is liquid metal and the dispersed phase is inert gas with very small electrical conductivity. This arrangement of phases is necessary in order to achieve significant MHD effect in the channel. In that case $\langle\tilde{\sigma}_d\rangle$ is negligible in comparison to $\langle\tilde{\sigma}_c\rangle$ and it can be shown that (Dobran[8]) the MHD generator performance parameters can be ascertained from two experimentally determined parameters $\tilde{K}_f = C_0 + \langle\tilde{w}_{dj}\rangle/\langle\tilde{J}\rangle$ and $\tilde{k}_c$. In the expression for $\tilde{K}_f$, $\langle\tilde{J}\rangle$ is the 2-phase volumetric flowrate per unit area; C_0 is the distribution parameter which accounts for the nonuniform flow and concentration profile in the channel (Zuber and Findlay[16]), and $\langle\tilde{w}_{dj}\rangle/\langle\tilde{J}\rangle$ accounts for the relative velocity between the phases. $\tilde{k}_c$ represents the effect of the nonuniform distribution of electrical conductivity with flow in the cross-sectional area of the channel and has been defined previously in this paper. Both $\tilde{K}_f$ and $\tilde{k}_c$ are functions of the 2-phase flow regime. When the applied magnetic field is perpendicular to the main flow direction in the MHD channel and when the channel divergence angle is small and the return current conductors properly routed, it can be shown (Dobran[8]) that the electric field and the induced current vectors in the fluid have only one component that is different from zero. This deduction follows from eqns (37)–(41). The induced magnetic field is in the direction parallel to the flow and becomes significant if the applied magnetic field is strong. With very strong applied magnetic fields, in the order of 10 Tesla, it is possible for the 2-phase flow to exhibit a significant transverse-to-the-main-flow pressure gradient which can cause redistribution of phases in the channel. This experimental fact is fully accounted for by the theory presented if *no* assumption is made that the pressure is uniform across the duct cross-sectional area. For details the reader is referred to Ref. [8]. The field strength of the order of 10 Tesla will exist also in magnetically confined plasma fusion reactors and the 2-phase flow will appear when the liquid metal is utilized as a primary blanket coolant. The boiling 2-phase flow of the liquid metal can be generated under fusion plasma quench conditions.

5. CONCLUDING REMARKS

The present work has presented sufficient conditions which give rise to consistency between different formulations of the 2-phase flow. It is shown that under rather general conditions the ergodic hypothesis on a finite averaging time interval can hold when applied to space-averaging operators. In order to explicitly determine a suitable averaging time interval, it is necessary to know the average power and the variance of the noise of space integrals. It would be of considerable value to undertake some experimental work for specific flow regimes in order to quantitatively identify the values of these parameters. The consistency conditions are expressed by eqns (19c, d) and (21c).

This paper has also extended the range of applicability of 2-phase flow models by bringing into the formulation electric and magnetic fields. In Section 2 of the paper it was demonstrated how various 2-phase flow models are constructed and in Section 4 an application to a simplified 1-dimensional Magnetohydrodynamic flow was carried out. The drift flux model formulation has been discussed for different MHD applications.

NOMENCLATURE

General

- a area
- B defined in Table 1
- $\mathbf{B}$ magnetic induction vector
- $\dot{c}$ speed of light in vacuum
- C duct boundary perimeter
- $\mathbf{E}$ electric field vector
- $E\{...\}$ expectation or ensemble average, $E\{\bar{\bar{z}}\} \equiv \int_{-\infty}^{\infty} \bar{\bar{z}}\, dP$
- $\bar{\bar{F}}$ stochastic integral defined by eqn (12)
- $\mathbf{g}$ gravitational force per unit mass
- h enthalpy
- $\Delta\tilde{h}_{dc}$ enthalpy difference, $\langle\tilde{h}_d\rangle - \langle\tilde{h}_c\rangle$
- $\mathbf{i}$ electric current density
- $\tilde{k}$ electrical-conductivity-flow correlation coefficient
- $\mathbf{J}$ center of volume velocity, defined by eqn (29)
- J defined in Table 1
- $\dot{m}$ mass flow-rate across the interface
- $\hat{n}$ unit normal vector
- $\bar{\bar{n}}$ noise stochastic process, defined by eqn (16)
- p static pressure
- P probability measure
- $\mathbf{q}$ heat flux vector
- $\mathbf{S}$ interface velocity vector
- t time
- T averaging time interval
- u internal energy
- v volume
- $\mathbf{V}$ velocity vector
- $\tilde{\mathbf{V}}_{dj}$ dispersed phase drift velocity, defined by eqn (30)
- w axial flow velocity in the duct
- $\mathbf{x}$ space vector
- z axial coordinate in the duct

Greek symbols

α void fraction
Γ mass generation rate per unit volume
δ probability sample function
Δ defined in Table 1
ϵ_0 dielectric permittivity of free space
$\bar{\bar{\eta}}$ stochastic correlation coefficient
λ defined in Table 2
Λ defined in Table 2
μ_0 magnetic permeability of free space
μ electron mobility
ν surface tension coefficient
ξ perimeter
$\boldsymbol{\pi}$ viscous stress tensor
$\boldsymbol{\pi}$ stress tensor, $-p\mathbf{I}+\boldsymbol{\pi}$
ρ mass density
σ electrical conductivity
$\sigma_{\bar{F}_{kT}}$ variance of stochastic process $\bar{\bar{F}}_{kT}$
$\sigma_{\partial \bar{F}_{kT}/\partial t}$ variance of stochastic process $\partial\bar{\bar{F}}_{kT}/\partial t$
$\sigma_{\bar{n}}$ variance of process $\bar{\bar{n}}$
Σ defined in Table 2
ϕ defined in Table 1
Φ defined in Table 2
I unit tensor
χ defined in Table 2
Ψ defined in Table 1
Ω defined in Table 2

Subscripts

c continuous phase
d dispersed phase
i interface between phases 1 and 2
k denotes phases d and c or 1 and 2
m mean value
S surface
w duct wall
T depends on the averaging time interval

Superscripts

T transpose

Special symbols

$\bar{\bar{\theta}}_k$ ensemble average, $E\{\bar{\bar{\theta}}_k\}$
$\langle\ \rangle$ area average operator, $\langle\ \rangle \equiv \frac{1}{a_k}\int_{a_k} \mathrm{d}a$

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(*Received 6 October 1980*)

APPENDIX A

Figure 1 illustrates 2-phase flow in a variable area channel. In the flow an infinitesimal control volume is considered that is bounded by the duct cross-sectional flow area and the duct boundaries. Taking the limit as $\Delta z \to 0$ the Leibniz and Gauss Divergence Theorems yield the following equations for a scalar field f, a vector field $\mathbf{F}$ and a tensor field F.

$$\int_{a_k(z,t)} \frac{\partial f}{\partial t}\,\mathrm{d}a = \frac{\partial}{\partial t}\int_{a_k(z,t)} f(\mathbf{x},t)\,\mathrm{d}a - \int_{\xi_i+C_k} f(\mathbf{x},t)\mathbf{S}_i\cdot\hat{n}_k \frac{\mathrm{d}\xi}{\hat{n}_k\cdot\hat{n}_{k\xi}} \tag{A1}$$

$$\int_{a_k(z,t)} \nabla\cdot\mathbf{F}\,\mathrm{d}a = \frac{\partial}{\partial z}\int_{a_k(z,t)} \hat{n}_z\cdot\mathbf{F}\,\mathrm{d}a + \int_{\xi_i+C_k} \hat{n}_k\cdot\mathbf{F}\frac{\mathrm{d}\xi}{\hat{n}_k\cdot\hat{n}_{k\xi}} \tag{A2}$$

$$\int_{a_k(z,t)} \nabla\times\mathbf{F}\,\mathrm{d}a = \frac{\partial}{\partial z}\int_{a_k(z,t)} \hat{n}_z+\mathbf{F}\,\mathrm{d}a\int_{\xi_i+C_k} \hat{n}_k\times\mathbf{F}\frac{\mathrm{d}\xi}{\hat{n}_k\cdot\hat{n}_{k\xi}} \tag{A3}$$

$$\int_{a_k(z,t)} \nabla\cdot\mathsf{F}\,\mathrm{d}a = \frac{\partial}{\partial z}\int_{a_k(z,t)} \hat{n}_z\cdot\mathsf{F}\,\mathrm{d}a + \int_{\xi_i+C_k} \hat{n}_k\cdot\mathsf{F}\frac{\mathrm{d}\xi}{\hat{n}_k\cdot\hat{n}_{k\xi}}. \tag{A4}$$

In these equations ξ_i is the liquid-vapor interface segment and C_k is the duct boundary segment for phase k.

APPENDIX B

Let $\bar{\bar{n}}_k$ be a covariance stationary normal stochastic process with zero mean and variance $\sigma_{\bar{n}_k}$. Then

$$E\{\bar{\bar{n}}_k(t)\bar{\bar{n}}_k(t+\tau)\} = R_{\bar{n}_k}(\tau) = R_{\bar{n}_k}(0)\,\mathrm{e}^{-\tau^2/2\sigma^2_{\bar{n}_k}}. \tag{B1}$$

The variance of eqn (16) is

$$\begin{aligned}\sigma^2_{\bar{F}_{kT}} &= \frac{1}{T^2}\int_{t-T/2}^{t+T/2}\int_{t-T/2}^{t+T/2} E\{\bar{\bar{n}}_k(t_1)\bar{\bar{n}}_k(t_2)\}\,\mathrm{d}t_1\,\mathrm{d}t_2\\ &= \frac{2}{T}\int_0^T\left(1-\frac{\tau}{T}\right)R_{\bar{n}_k}(0)\,\mathrm{e}^{-\tau^2/2\sigma^2_{\bar{n}_k}}\,\mathrm{d}\tau\\ &= R_{\bar{n}_k}(0)\left[\sqrt{2\pi}\left(\frac{\sigma_{\bar{n}_k}}{T}\right)\mathrm{erf}\left(\frac{T}{\sqrt{2}\sigma_{\bar{n}_k}}\right)+2\left(\frac{\sigma_{\bar{n}_k}}{T}\right)^2\mathrm{e}^{-T^2/2\sigma^2_{\bar{n}_k}}-1)\right]\end{aligned} \tag{B2}$$

and the variance of eqn (19b) is

$$\begin{aligned}\sigma^2_{\partial\bar{F}_{kT/\partial t}} &= \frac{1}{T^2}\int_{t-T/2}^{t+T/2}\int_{t-T/2}^{t+T/2} E\left\{\frac{\partial\bar{\bar{n}}_k}{\partial t_1}\frac{\partial\bar{\bar{n}}_k}{\partial t_2}\right\}\mathrm{d}t_1\,\mathrm{d}t_2\\ &= -\frac{2}{T}\int_0^T\left(1-\frac{\tau}{T}\right)\frac{\mathrm{d}^2R_{\bar{n}_k}(\tau)}{\mathrm{d}\tau^2}\,\mathrm{d}\tau\\ &= \frac{2}{T^2}R_{\bar{n}_k}(0)(1-\mathrm{e}^{-T^2/2\sigma^2_{\bar{n}_k}})\end{aligned} \tag{B3}$$

The power spectrum of the process $\bar{\bar{n}}_k$ is

$$S(w) = \int_{-\infty}^{\infty}\mathrm{e}^{-jwt}R_{\bar{n}_k}(t)\,\mathrm{d}t$$

from which it follows by the Fourier Inversion Theorem that

$$R_{\bar{n}_k}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}S(w)\,\mathrm{d}w.$$

Thus, $R_{\bar{n}_k}(0)$ is the *average power* of the process $\bar{\bar{n}}_k$.

With finite average power, from eqns (B2) and (B3) we have

1.
$$\lim \sigma_{\bar{F}_{kT}} = 0$$
$$\frac{\sigma_{\bar{n}_k}}{T}\to 0$$
$$P_1\{|\bar{\bar{F}}_{kT}-E\{\bar{\bar{F}}_{kT}\}|<\epsilon_1\} = 1,\quad \epsilon_1>0.$$

2.
$$\lim \sigma_{\partial\bar{F}_{kT}/\partial t} = 0$$
$$\frac{\sigma_{\bar{n}_k}}{T}\to 0 \ \text{and}\ \frac{\sqrt{R_{\bar{n}_k}(0)}}{T}\to 0$$
$$P_2\left\{\left|\frac{\partial\bar{\bar{F}}_{kT}}{\partial t}-\frac{\partial}{\partial t}E\{\bar{\bar{F}}_{kT}\}\right|<\epsilon_2\right\} = 1,\quad \epsilon_2>0.$$