

ON THE FORMULATION OF CONSERVATION, BALANCE AND CONSTITUTIVE EQUATIONS FOR  
MULTIPHASE FLOWS

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ABSTRACT

The formulation of conservation, balance and constitutive equations for multiphase flows requires the considerations of the levels of description of the theory and of the existence and consistency of the theoretical structure in a physical world. A theory of multiphase mixtures should be formulated on a level that avoids the solution of the multiboundary problem, and it should be consistent with proper and rigorous averaging procedures applied to the macroscopic field equations of each phase. The averaging or reductionistic procedure cannot, however, yield a rigorous theory, for, eventually, this must be abandoned in order to define rigorously kinematic concepts. The degree to which a reductionistic or a holistic theoretical structure is accepted plays a crucial role in the development of constitutive equations, and both of these approaches are discussed in this paper. Based on these investigations, results for constitutive equations are reviewed, and it is shown that although the holistic or continuum mechanics approach yields a self-consistent theory and many useful results, it is not very clear that this procedure will yield a theory of multiphase mixtures that is fully consistent with the physical world.

1. INTRODUCTION

While a considerable advance has been made in the development of useful models for two-phase flows, this development has not been as rapid for flows involving three or more phases. The main reasons for this are: 1) two-phase flows are easier to study analytically and experimentally, and 2) the practical need to understand two-phase flows has been greater. Modeling of transient and nonequilibrium flow phenomena places severe demands on any model of multiphase flows, and for this reason it is, therefore, necessary that the conservation, balance and constitutive equations have a strong foundation: pragmatic in order to conform with the physical reality and idealistic in order to possess a sound theoretical structure. The degree of pragmatism or idealism in a model is always disputable which, of course, must be in order to advance the present state of knowledge.

The analysis of multiphase flows is difficult due to their finite interfacial area. Most, if not all, of the present models reflect a particular special case of this area or flow regime. Flow regime modeling is very useful for steady flows; however, many practical situations involve unsteady multiphase flows for which the current modeling practices are highly unsatisfactory due to the inadequate knowledge of the flow regime transition physics. Although it is conceptually possible to describe a multiphase mixture by the

well-established macroscopic conservation and balance equations of each phase separately and by the interfacial boundary conditions, it is, however, not possible practically to follow the usually very large number of interfaces, and, as a consequence, the analytic problem becomes intractable. The recognition of this fact has led to the diverse modeling practices - all of which are based on a continuum approach. The conservation and balance equations of multiphase flows constructed in this manner are either postulated [1-2] or they are obtained by an application of suitable averaging of the macroscopic field equations of each phase [3-8]. Both approaches have deficiencies: the first, because it lacks a strong physical justification, and the second, because it lacks a reconciliation between different levels of description. The above-mentioned idealistic or pragmatic approaches which are followed in the construction of conservation and balance equations for multiphase flows also play a crucial role in the subsequent development of constitutive equations, as discussed below.

In view of the fundamental issues which surround the formulation of conservation, balance and constitutive equations for multiphase mixtures, it is important that these issues be discussed in order that future theoretical developments be built on a stronger foundation. This discussion is important since all too often many proposed models tend to become personalized theories adopted by those who propose them and by the users of these theories who become familiar with them. Wallis [9], in a recent review article, concludes that "The ad hoc approach is likely to retain its preeminence as the dominant method of analysis for the foreseeable future." This conclusion, while being pessimistic, is nevertheless well-founded since it is based on extensive experiences in two-phase flow modeling. With a considerable amount of knowledge already existing and with the availability of computational techniques and advanced instrumentation, the future of multiphase flow modeling appears more optimistic to this author.

The formulation of conservation and balance equations for multiphase flows is discussed in Section 2 where it is shown that despite a considerable progress many problems still remain. In Section 3, the discussion is carried out on the formulation of constitutive equations where different methods of analysis are reviewed. It is shown there that it is not very clear which approach to utilize but that the holistic or continuum mechanics procedure provides very useful results.

## 2. ON THE FORMULATION OF CONSERVATION AND BALANCE EQUATIONS FOR MULTIPHASE FLOWS

All the formulations of conservation and balance equations for multiphase flows utilize continuum modeling which are based on either of the two following approaches:

- a) Extension of single-phase multicomponent mixture equations to multiphase mixtures.
- b) Averaging of the instantaneous field equations of each phase over the appropriate time or space segments.

In the first approach, the theory of single-phase multicomponent mixtures [10,11] is extended to multiphase mixtures [1,12]. The fundamental assumptions in this theory are the coexistence of all phases at each point in space and that each coexisting phasic continuum can be assigned unique properties such as density, velocity, energy, etc. Additional important characteristics

of this theory are as follows:

- 1) No attempt is made in the theory to justify the field equations from any molecular or local macroscopic level for each phase.
- 2) The conservation and balance equations for the mixture as a whole do not allow for source terms.
- 3) In the absence of external torques or couple stresses, the stress tensor for the mixture as a whole is symmetric.
- 4) The entropy equation for the mixture as a whole is assumed to be valid unless the nonequality of entropy and heat fluxes is assumed in which case an entropy equation for each constituent in the mixture is also postulated to be valid.

While in the theory of single-phase multicomponent mixtures the last three conditions given above appear to be justifiable in so far as no contradictions with classical equilibrium thermodynamics have been found, it is not clear that these conditions should also be valid for a multiphase mixture - at least as long as we accept the validity of averaging procedures discussed below.

The averaging approach followed in the construction of conservation and balance equations for multiphase mixtures is based on the concept that a proper and clear way to obtain multiphase field equations is to perform averaging of the local macroscopic field equations for each phase over a suitable time or space domain. Thus, Drew [3] carried out time and volume averaging, Ishii [4] performed time averaging, Delhaye and Achard [5] derived one-dimensional volume-, time-, area- and space-averaged two-phase flow equations, Dobran [7] derived multidimensional volume-averaged equations, Bataille and Kestin [6] and Marle [8] utilized the theory of distributions to obtain volume and volume-time-averaged multiphase field equations, respectively.

With all these averaging procedures proliferating through the literature, some doubts arise whether any of them should be accepted as yielding the correct form of multiphase field equations. While ad hoc models can be justified on the basis of a particular averaging procedure, it is important to note that the objective of averaging is to produce a rigorous set of conservation and balance equations for multiphase mixtures. This being the case, then, is there any relative merit between different averaging approaches? In the following, we shall attempt to reconcile the relative advantages and disadvantages of various averaging formulations.

The time-, area- and segment-averaged equations have the disadvantage of being singular under certain flow conditions (for the case of time-averaged formulation when the interface becomes stationary in the flow field). This disadvantage is, clearly, bothersome and to smooth out the singularities, further averaging is usually performed. The smoothed field equations are all of space-time variety and immediately bring about the following concerns:

- 1) The field equations do not lend themselves to the investigation of constitutive equations by utilizing the modern foundations of continuum mechanics since the principle of objectivity or material frame invariance cannot be consistently utilized on the time-averaged equations unless they are also assumed to be ergodic.
- 2) Averaging over a restricted space domain to obtain one-dimensional field equations leads to difficulty when studying the constitutive

equations since the conclusions about the invariance properties can only be obtained from the general three-dimensional equations and not from any restricted class of them.

- 3) The time-averaged turbulent field equations are different from the space-averaged equations in the sense that the latter equations contain additional turbulent correlations, as discussed below.

The nature and subtleties involved in different averaging procedures discussed above can be expounded by selecting one such procedure for discussion and then by pointing out the advantages and disadvantages of this procedure and its relation to other averaging approaches. For this purpose, the multi-dimensional volume averaging procedure developed by Dobran [7] is selected.

The formulation of conservation and balance equations for multiphase flows utilizing the volume averaging procedure follows by considering a multiphase flow field such as that shown in Fig. 1. The volume pertaining to phase  $\alpha$ ,  $\alpha = 1, \dots, \gamma$ , in the arbitrary and fixed averaging volume  $V$  is  $V_\alpha = \sum_{\delta} V_{\alpha\delta}$  such that  $V = \sum_{\alpha=1}^{\gamma} V_\alpha$ . For each phase  $\alpha$  and volume  $V_{\alpha\delta}$  the macroscopic field equations representing the conservation of mass and balance of linear momentum, angular momentum, energy and entropy can be written as follows [7,13]:

$$\frac{\partial}{\partial t} (\rho_{\alpha\delta} \Psi_{\alpha\delta}) + \nabla \cdot (\rho_{\alpha\delta} \Psi_{\alpha\delta} \mathbf{v}_{\alpha\delta}) + \nabla \cdot \mathbf{J}_{\alpha\delta} - \rho_{\alpha\delta} \Phi_{\alpha\delta} - \rho_{\alpha\delta} B_{\alpha\delta} = 0 \quad (1)$$

where  $\Psi_{\alpha\delta}$ ,  $\mathbf{J}_{\alpha\delta}$ ,  $\Phi_{\alpha\delta}$  and  $B_{\alpha\delta}$  are given in Table 1. At the interface between phases  $\alpha$  and  $\beta$  we have the jump conditions [7,13]:

$$(m_{\alpha\delta} \Psi_{\alpha\delta} + J_{\alpha\delta} \mathbf{n}_{\alpha\delta}) + (m_{\beta\eta} \Psi_{\beta\eta} + J_{\beta\eta} \mathbf{n}_{\beta\eta}) = \Delta_{\alpha\delta} \quad (2)$$

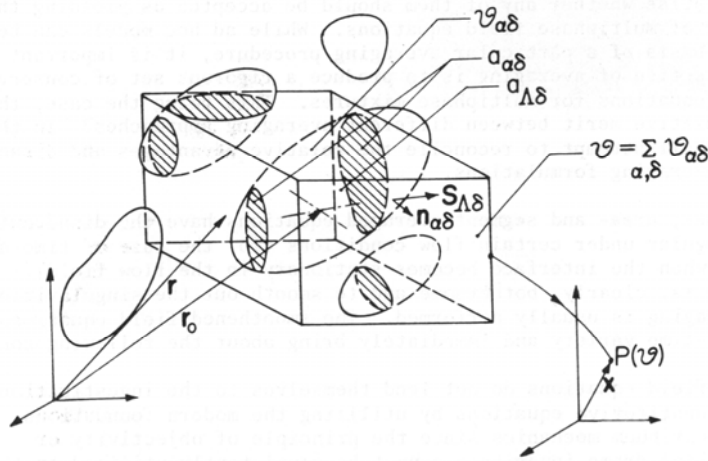


Figure 1. Definition of the Averaging Volume in a Multiphase Flow Field

CONSERVATION OF	$\Psi_{\alpha\delta}$	$\mathbf{J}_{\alpha\delta}$	$\Phi_{\alpha\delta}$	$B_{\alpha\delta}$	$\Delta_{\alpha\delta}$
MASS	1	0	0	0	0
LINEAR MOMENTUM	$\mathbf{v}_{\alpha\delta}$	$-\mathbf{T}_{\alpha\delta}$	$\mathbf{b}_{\alpha\delta}$	0	$(\Delta_m)_{\alpha\delta} = (2H\nu\mathbf{n} + \nabla_s \nu)_{\alpha\delta}$
ANGULAR MOMENTUM	$\mathbf{r} \wedge \mathbf{v}_{\alpha\delta}$	$-\mathbf{r} \wedge \mathbf{T}_{\alpha\delta}$	$\mathbf{r} \wedge \mathbf{b}_{\alpha\delta}$	0	$\mathbf{r} \wedge (\Delta_m)_{\alpha\delta}$
ENERGY	$\epsilon_{\alpha\delta} + \frac{\mathbf{v}_{\alpha\delta}^2}{2}$	$\mathbf{q}_{\alpha\delta} - \mathbf{T}_{\alpha\delta}^T \mathbf{v}_{\alpha\delta}$	$\mathbf{b}_{\alpha\delta} \cdot \mathbf{v}_{\alpha\delta}$	$r_{\alpha\delta}$	$(\Delta_\epsilon)_{\alpha\delta} = (2H\nu\mathbf{n} \cdot \mathbf{S} + \nabla_s \nu \cdot \mathbf{S} + \nu \nabla_s \cdot \mathbf{S})_{\alpha\delta}$
ENTROPY	$s_{\alpha\delta}$	$\frac{h_{\alpha\delta}}{\Theta_{\alpha\delta}}$	$\frac{r_{\alpha\delta}}{\Theta_{\alpha\delta}}$	$\xi_{\alpha\delta} \geq 0$	$(\Delta_s)_{\alpha\delta} \geq 0$

Table 1. Coefficients of the Conservation Equations (1) and (2)

where

$$m_{\alpha\delta} = \rho_{\alpha\delta} (\mathbf{v}_{\alpha\delta} - \mathbf{S}_{\Lambda\delta}) \cdot \mathbf{n}_{\alpha\delta} \quad (3)$$

$\mathbf{S}_{\Lambda\delta}$  is the interface velocity,  $\mathbf{n}_{\alpha\delta}$  is the unit normal vector which points outwards from the volume  $V_{\alpha\delta}$ ,  $m_{\alpha\delta}$  is the interphase mass transfer rate which arises due to the phase change, and  $\Delta_{\alpha\delta}$  is the surface tension term (or some other interfacial source density) and is defined in Table 1.  $H$  is the mean interface curvature,  $\nabla_s$  is the surface gradient vector and  $\nu$  is the surface tension coefficient. Clearly,  $\alpha \neq \beta$ ,  $\mathbf{n}_{\alpha\delta} = -\mathbf{n}_{\beta\eta}$  and  $\mathbf{S}_{\Lambda\delta} = \mathbf{S}_{\beta\eta}$ . Furthermore in Table 1,  $\mathbf{v}_{\alpha\delta}$  is the velocity,  $\rho_{\alpha\delta}$  is the mass density,  $\mathbf{T}_{\alpha\delta}$  is the stress tensor,  $\mathbf{b}_{\alpha\delta}$  is the external body force per unit mass,  $\epsilon_{\alpha\delta}$  is the internal energy per unit mass,  $\mathbf{q}_{\alpha\delta}$  is the heat flux vector,  $s_{\alpha\delta}$  is the entropy per unit mass,  $h_{\alpha\delta}$  is the entropy flux vector,  $\Theta_{\alpha\delta}$  is the absolute temperature,  $r_{\alpha\delta}$  is the heat generation rate per unit volume, and  $\xi_{\alpha\delta}$  and  $\Delta_s$  are the entropy production rates in the continuum and at the interface, respectively.

The volume averaging process consists of volume averaging of Eq. (1) over the portion of volume  $V$  which the phase  $\alpha$  occupies at time  $t$ , i.e.

$$\sum_{\delta} \int_{V_{\alpha\delta}} [\text{Eq. (1)}] dV = 0 \quad (4)$$

Utilizing Leibnitz's rule and Divergence Theorem [7], Eq. (4) is reduced to:

$$\begin{aligned} \frac{\partial}{\partial t} V_\alpha \langle \rho_\alpha \Psi_\alpha \rangle + \nabla \cdot V_\alpha \langle \rho_\alpha \Psi_\alpha \mathbf{v}_\alpha \rangle + \nabla \cdot V_\alpha \langle \mathbf{J}_\alpha \rangle - V_\alpha \langle \rho_\alpha \Phi_\alpha \rangle - V_\alpha \langle \rho_\alpha B_\alpha \rangle = \\ - \sum_{\delta} \int_{a_{\Lambda\delta}} (m_{\alpha\delta} \Psi_{\alpha\delta} + J_{\alpha\delta} \mathbf{n}_{\alpha\delta}) da, \end{aligned} \quad (5)$$

where the volume-averaged quantities are defined as follows:

$$\langle F_\alpha \rangle \equiv \frac{1}{V_\alpha} \sum_{\delta} \int_{V_{\alpha\delta}} F_{\alpha\delta} dV \quad (6)$$

Equation (5) is the general transport equation of phase  $\alpha$  and will be identified as the macro-macroscopic transport equation of phase  $\alpha$  as opposed to Eq. (1) which is the macroscopic transport equation of phase  $\alpha$ . The volume averaging procedure constructed above induces a mapping such that Eq. (5) represents the superposition of  $\gamma$  continua, i.e., at each point of space and at each instant of time a continuum of phase  $\alpha$  can be identified and in this report the theory of multiphase mixtures constructed in this manner is identical to the theory of single-phase multicomponent mixtures.

All space and time averaging procedures lead to an equation similar to Eq. (5); the difference between them, however, is in the interpretation of the averaged quantities. Furthermore, Eq. (5) is not useful until a connection is made between the variables defined in the macroscopic field Eq. (1) ( $\rho_\alpha$ ,  $\psi_\alpha$ ,  $\mathbf{v}_\alpha$ , etc.) and the averaged variables in the macro-macroscopic field Eq. (5). In the next step of the analysis, therefore, two kinds of averaged variables are introduced. These are

i) The density-weighted averages,  $\tilde{H}_\alpha$

$$\tilde{H}_\alpha \equiv \frac{\langle \rho_\alpha H_\alpha \rangle}{\langle \rho_\alpha \rangle} \quad (7)$$

ii) The phase averages,  $\bar{H}_\alpha$  and  $\bar{\bar{H}}_\alpha$

$$\bar{H}_\alpha \equiv \frac{V_\alpha}{V} \langle H_\alpha \rangle \quad (8)$$

$$\bar{\bar{H}}_\alpha \equiv \langle H_\alpha \rangle \quad (9)$$

The density-weighted variables are defined for all those macroscopic variables on a unit mass basis. In addition, the macro-macroscopic velocity of phase  $\alpha$ ,  $\tilde{\mathbf{v}}_\alpha$ , is also defined as the density-weighted average for the reasons that a finite size instrument can only measure this velocity and, when substituting for  $\psi_\alpha$ ,  $J_\alpha$ ,  $\phi_\alpha$  and  $B_\alpha$  from Table 1 for the conservation of mass in Eq. (5), the conservation of mass equation of phase  $\alpha$ ,

$$\frac{\partial}{\partial t} \bar{\rho}_\alpha + \nabla \cdot \bar{\rho}_\alpha \tilde{\mathbf{v}}_\alpha = -\sum_{\delta} \int_{a_{\Lambda\delta}} m_{\alpha\delta} da \quad , \quad (10)$$

becomes similar in appearance to the macroscopic continuity equation of the same phase. In the time-averaged formulation, the velocity  $\tilde{\mathbf{v}}_\alpha$  is defined as the time-averaged velocity and in the volume-time-averaged formulation it is defined over a space-time domain.

All space and time averaging approaches lead to the conservation of mass equation that is similar to Eq. (10); however, the balance equations of momentum, energy and entropy differ from one formulation to another in the interpretation of the following term:

$$V_\alpha \langle \rho_\alpha \psi_\alpha \mathbf{v}_\alpha \rangle \quad (11)$$

In the time-averaged formulation, the macroscopic field variable  $\psi_\alpha$  is decomposed into a density-weighted time average  $\tilde{\psi}_\alpha$  and into a fluctuating or turbulent field  $\psi'_\alpha$ , i.e.

$$\psi_\alpha = \tilde{\psi}_\alpha + \psi'_\alpha \quad , \quad (12)$$

whereas in the volume and time-averaged formulation  $\tilde{\psi}_\alpha$  in Eq. (12) must be interpreted as the density-weighted space-time average and  $\psi'_\alpha$  as the space time fluctuation. In the volume-averaged formulation, no splitting up of the field variables as defined by Eq. (12) is performed; instead, the covariance coefficients are introduced as the differences between the average of the product of macroscopic variables and the product of the macro-macroscopic field variables, i.e.,

$$V_\alpha \langle \rho_\alpha \psi_\alpha \mathbf{v}_\alpha \rangle - V \bar{\rho}_\alpha \tilde{\psi}_\alpha \tilde{\mathbf{v}}_\alpha \quad (13)$$

From the above discussion, it is clear that depending on the averaging procedure used in the formulation of conservation equation of mass and balance equations of momentum, energy and entropy for multiphase flows, the resulting field equations can be quite different. Using Eq. (12) in the time- and space and time-averaged formulations gives rise to the macro-macroscopic field equations in terms of mean and fluctuating quantities where the latter quantities are interpreted as the turbulent contributions. The volume-averaged macro-macroscopic field equations [7] contain instead the covariance coefficients expressed by Eq. (13) where they are interpreted as the nonlocal effect in the theory. This interpretation arises from the effect of the size of averaging volume  $V$  on these coefficients. When the multiphase mixture becomes dispersed the effect of nonlocality disappears.

Another significant difference between the time-averaged and volume-averaged field equations is in the formulation of turbulence characteristics. Time averaging of the volume-averaged linear momentum equation, for example, gives rise to velocity-velocity, partial density-velocity, partial density-velocity-velocity and partial density-pressure correlations while the time-averaged linear momentum equation gives rise to the velocity-velocity correlations [4] only.

In view of the discussion above, it is my conclusion that the volume-averaged conservation and balance equations for multiphase mixtures have the most desirable properties to motivate the construction of a theory of multiphase flows since:

1. The equations are not singular.
2. The equations are amenable to the study of invariance properties using the principle of objectivity or frame indifference without giving rise to any inconsistencies.
3. The turbulent field equations are richer.

In the construction of a theory of multiphase mixtures utilizing the averaging approaches described above, one follows a reductionistic or pragmatic approach. This means that the theory should be formulated by piecing together its fundamental building blocks (the macroscopic field equations of each phase) with the pieces which describe how these blocks should be put together (the macroscopic interface or boundary conditions). Clearly, different averaging procedures lead to different interpretations of the resulting field equations, as discussed above, and some procedures are more desirable than others (for example, the volume averaging) since the multiphase equations constructed in this way insure the proper existence and consistency properties of the theory. This, however, should not be viewed necessarily as a proper way of constructing a rigorous theory, for if this were the case, then a reconciliation would have to be made between the macroscopic and macro-macroscopic field variables. How this can be done in a physically convincing manner is at the present time open to speculations.

The approach that has been followed in constructing a theory of multiphase mixtures in Reference 7 involves utilizing the volume averaging procedure not in the hope of deriving the multiphase field equations but to motivate the construction of the theory since, as discussed above, the volume-averaged equations have more desirable properties than the equations from other averaging approaches. Having established the macro-macroscopic transport Eq. (5) and definitions expressed by Eqs. (6)-(9), it is assumed that to each coexisting or superimposed continuum of phase  $\alpha$  can be assigned particles  $X_\alpha$  and configurations  $\chi_\alpha$  of these particles at time  $t$  such that at the current space position  $\mathbf{x}$  we can write

$$\mathbf{x} \equiv \chi_\alpha(X_\alpha, t) \quad (14)$$

Furthermore, to each particle of phase  $\alpha$  can be assigned a reference configuration  $\kappa_\alpha$  at some time  $t_0 < t$  such that its reference position is expressed by the following equation:

$$X_\alpha \equiv \kappa_\alpha(X_\alpha, t_0) \quad (15)$$

Combining Eqs. (14) and (15) we obtain:

$$\mathbf{x} = \chi_\alpha(\kappa_\alpha^{-1}(X_\alpha, t_0), t) \equiv \chi_{\alpha\kappa}(X_\alpha, t) \quad (16)$$

where  $\chi_{\alpha\kappa}$  is the deformation function of a particle of phase  $\alpha$ . The velocity and acceleration of particle  $\chi_{\alpha\kappa}$  follow by differentiating Eq. (16) with respect to time while keeping the identity of particle constant. In the next crucial step of the analysis, the velocity and acceleration obtained from Eq. (16) are identified with the density-weighted velocity and acceleration defined by Eq. (7), i.e.,

$$\tilde{\mathbf{v}}_\alpha \equiv \frac{\partial \chi_{\alpha\kappa}(X_\alpha, t)}{\partial t} \quad (17)$$

$$\dot{\tilde{\mathbf{v}}}_\alpha \equiv \frac{\partial^2 \chi_{\alpha\kappa}(X_\alpha, t)}{\partial t^2} \quad (18)$$

It should be noted in Eq. (18) and in the equations that follow that a backward prime affixed to the subscripted symbols with  $\alpha$  indicates the material derivative 'following' the particle of the phase  $\alpha$ .

The density of a multiphase mixture,  $\rho$ , and the center of mass velocity,  $\mathbf{v}$ , are defined respectively as

$$\rho \equiv \sum_{\alpha=1}^{\gamma} \bar{\rho}_\alpha, \quad \rho \mathbf{v} \equiv \sum_{\alpha=1}^{\gamma} \bar{\rho}_\alpha \tilde{\mathbf{v}}_\alpha \quad (19)$$

while the diffusion velocity of phase  $\alpha$ ,  $\mathbf{u}_\alpha$ , is defined by

$$\mathbf{u}_\alpha \equiv \tilde{\mathbf{v}}_\alpha - \mathbf{v} \quad (20)$$

If  $\tilde{\Gamma}_\alpha$  and  $\Gamma$ , where  $\rho\dot{\Gamma} \equiv \sum_{\alpha=1}^{\gamma} \bar{\rho}_\alpha \dot{\tilde{\Gamma}}_\alpha$ , are differentiable functions of  $\mathbf{x}$  and  $t$  then by utilizing the definitions of material derivatives

$$\dot{\tilde{\Gamma}}_\alpha \equiv \frac{\partial \tilde{\Gamma}_\alpha}{\partial t} + (\nabla \tilde{\Gamma}_\alpha) \tilde{\mathbf{v}}_\alpha, \quad \dot{\Gamma} \equiv \frac{\partial \Gamma}{\partial t} + (\nabla \Gamma) \mathbf{v} \quad (21)$$

we have:

$$\rho\dot{\Gamma} = \sum_{\alpha=1}^{\gamma} [\bar{\rho}_\alpha \dot{\tilde{\Gamma}}_\alpha - \nabla \cdot \bar{\rho}_\alpha \tilde{\Gamma}_\alpha \mathbf{u}_\alpha + (\frac{\partial \bar{\rho}_\alpha}{\partial t} + \nabla \cdot \bar{\rho}_\alpha \tilde{\mathbf{v}}_\alpha) \tilde{\Gamma}_\alpha - (\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}) \Gamma_\alpha], \quad (22)$$

where  $\partial \rho / \partial t + \nabla \cdot \rho \mathbf{v} = 0$  by virtue of the conservation of mass equation for the mixture as a whole (Eq. (A.1)).

Utilizing Eqs. (6) - (9) and (17) - (21) in Eq. (5) results in the following conservation and balance equations for phase  $\alpha$ :

$$\text{Mass:} \quad \dot{\tilde{\rho}}_\alpha + \bar{\rho}_\alpha \nabla \cdot \tilde{\mathbf{v}}_\alpha = \hat{c}_\alpha \quad (23)$$

$$\text{Linear Momentum:} \quad \bar{\rho}_\alpha \dot{\tilde{\mathbf{v}}}_\alpha = \nabla \cdot \bar{\mathbf{T}}_\alpha + \bar{\rho}_\alpha \tilde{\mathbf{b}}_\alpha + \hat{\mathbf{p}}_\alpha \quad (24)$$

$$\text{Angular Momentum:} \quad \hat{\mathbf{M}}_\alpha = \bar{\mathbf{T}}_\alpha - \bar{\mathbf{T}}_\alpha^T \quad (25)$$

$$\text{Energy:} \quad \bar{\rho}_\alpha \dot{\tilde{\epsilon}}_\alpha = \text{tr}(\bar{\mathbf{T}}_\alpha^T \nabla \tilde{\mathbf{v}}_\alpha) - \nabla \cdot \bar{\mathbf{q}}_\alpha + \bar{\rho}_\alpha \tilde{\mathbf{r}}_\alpha + \hat{\epsilon}_\alpha \quad (26)$$

$$\text{Entropy:} \quad \bar{\rho}_\alpha \dot{\tilde{s}}_\alpha + \nabla \cdot (\frac{\bar{\mathbf{q}}_\alpha}{\bar{\theta}_\alpha}) - \frac{\bar{\rho}_\alpha \tilde{\mathbf{r}}_\alpha}{\bar{\theta}_\alpha} + \hat{c}_\alpha \tilde{s}_\alpha + \hat{s}_\alpha \geq 0 \quad (27)$$

In Eqs. (23) - (27),  $\hat{c}_\alpha$  is the mass supply of phase  $\alpha$  or the rate of phase change per unit volume of the mixture,  $\bar{\mathbf{T}}_\alpha$  is the (partial or phase average) stress tensor,  $\tilde{\mathbf{b}}_\alpha$  is the external body force,  $\hat{\mathbf{p}}_\alpha$  is the linear momentum supply,  $\hat{\mathbf{M}}_\alpha$  is the angular momentum supply,  $\tilde{\epsilon}_\alpha$  is the internal energy,  $\bar{\mathbf{q}}_\alpha$  is the (partial or phase average) heat flux vector,  $\tilde{\mathbf{r}}_\alpha$  is the heat generation rate,  $\hat{\epsilon}_\alpha$  is the energy supply,  $\tilde{s}_\alpha$  is the entropy,  $\hat{s}_\alpha$  is the entropy supply and  $\bar{\theta}_\alpha$  is the absolute temperature.  $\bar{\mathbf{T}}_\alpha$  denotes the transpose of the second order tensor  $\bar{\mathbf{T}}_\alpha$  (a linear transformation), and  $\text{tr}(\bar{\mathbf{T}}_\alpha^T \nabla \tilde{\mathbf{v}}_\alpha)$  denotes the trace of the product of two linear transformations  $\bar{\mathbf{T}}_\alpha$  and  $\nabla \tilde{\mathbf{v}}_\alpha$ , i.e.  $\text{tr}(\bar{\mathbf{T}}_\alpha^T \nabla \tilde{\mathbf{v}}_\alpha) = \bar{T}_{\alpha ij} \partial \tilde{v}_{\alpha j} / \partial x_i$ . The complete expressions for the supply terms  $\hat{c}_\alpha$ ,  $\hat{\mathbf{p}}_\alpha$ ,  $\hat{\mathbf{M}}_\alpha$ ,  $\hat{\epsilon}_\alpha$  and  $\hat{s}_\alpha$  are presented in Reference 7. Here, only the expressions for mass supply  $\hat{c}_\alpha$  and linear momentum supply  $\hat{\mathbf{p}}_\alpha$  are shown. Thus:

$$\hat{c}_\alpha \equiv -\frac{1}{V} \sum_{\delta} \int_{a_{\Lambda\delta}} m_{\alpha\delta} da \quad (28)$$

$$\hat{\mathbf{p}}_\alpha \equiv -\hat{c}_\alpha \tilde{\mathbf{v}}_\alpha - \frac{1}{V} \sum_{\delta} \int_{a_{\Lambda\delta}} (m_{\alpha\delta} \mathbf{v}_{\alpha\delta} - \mathbf{T}_{\alpha\delta} \mathbf{n}_{\alpha\delta}) da - \nabla \cdot \mathbf{C}_{1\alpha}, \quad (29)$$

where

$$\mathbf{C}_{1\alpha} \equiv \frac{V_\alpha}{V} \langle \rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha \rangle - \bar{\rho}_\alpha \tilde{\mathbf{v}}_\alpha \otimes \tilde{\mathbf{v}}_\alpha \quad (30)$$

is the covariance coefficient that represents the nonlocal effect and the symbol  $\otimes$  denotes the tensor product.

The conservation and balance equations for a multiphase mixture are obtained by summing up from  $\alpha = 1$  to  $\alpha = \gamma$  each of the Eqs. (23) - (27). The resulting field equations as well as the compatibility conditions between the

phasic and mixture equations are reproduced from Reference 7 in Appendix A of this paper for completeness.

In the volume-averaged equations presented above, the covariance coefficients in the momentum, energy and entropy supplies or interaction terms of phase  $\alpha$  can be viewed as accounting in the theory for nonlocal effects. When the multiphase mixture becomes dispersed and the averaging volume approaches zero the nonlocal effects should be negligible. With a finite interfacial area the region of nonlocality can be associated with the size of the averaging volume  $V$  and it is through such a construction that  $V$  can reenter into the framework of the theory.

The macro-macroscopic multiphase field equations (Eqs. (23) - (27)) are similar in appearance to the single-phase multicomponent mixture equations [10,11] except for the following features:

- 1) In the multiphase field equations above, the linear momentum source term  $\hat{p}_m$ , the angular momentum source term  $\hat{M}_m$ , the energy source term  $\hat{e}_m$  and the entropy source term  $\hat{s}_m$  are not necessarily equal to zero since they account for two effects: a) for the interfacial sources such as surface tension, and b) for the nonlocal effects.
- 2) The stress tensor for the mixture as a whole is not necessarily symmetric due to a) and b) in 1) above even if the mixture is not acted upon by the body couples and couple stresses.
- 3) In the above theory of multiphase mixtures, the entropy Eq. (27) for phase  $\alpha$  appears naturally while in the theory of multicomponent mixtures [10,11] an entropy equation for the mixture as a whole can only be used in order to achieve consistency of the theory with the classical equilibrium thermodynamics. From this, it follows that if each phase is composed of two or more chemical constituents, then an entropy equation for each constituent in the phase is not allowed.

As the above discussion shows, it should be clear that the final set of field equations for multiphase flows, Eqs. (23) - (27), are not entirely formulated utilizing a reductionistic approach, for this would not have led to the rigorous theory as brought about by the definition of kinematic concepts in Eqs. (14) - (22). The final set of field equations reflect a blend of pragmatism as well as idealism: pragmatism in the sense that these equations are consistent with the volume averaging approach and idealism in the sense that the equations are given a theoretical foothold by the definition of kinematic concepts. We will see in the next section that the degree to which reductionistic or holistic belief is placed on the models has a great bearing on the investigation of constitutive equations for multiphase mixtures.

### 3. ON CONSTITUTIVE EQUATIONS FOR MULTIPHASE FLOWS

In the previous section, we have encountered some methods for formulating field equations for multiphase mixtures although, strictly speaking, there are only two such methods that are exclusively utilized: extension of the single-phase multicomponent mixture field equations to the multiphase mixtures and averaging of the macroscopic field equations of each phase over the appropriate time and space segments. A similar situation also exists in the construction of constitutive equations for multiphase flows in the sense that use is made of one of the following two general approaches:

- a) Formulation of constitutive equations from the macroscopic modeling considerations (reductionistic approach).
- b) Formulation of constitutive equations from macro-macroscopic modeling considerations (holistic approach).

In the first approach, where the constitutive equations are deduced through the macroscopic modeling, two procedures can be distinguished: 1) averaging of the macroscopic constitutive equations of each phase over the appropriate time or space segments [4], and 2) phenomenological approach inspired by the flow-pattern modeling [6,9, workers carrying out ad hoc modeling of multiphase flows].

An attempt to deduce the constitutive equations for two-phase flows by time averaging of the macroscopic constitutive equations of each phase was made by Ishii [4]. This reductionistic approach can be criticized for the following deficiencies:

- 1) Failure to account in the constitutive equations for the organized structure of the multiphase flow field. The degree of this deficiency can, of course, be disputed but to justify it, consider a bubbly two-phase flow. At the macroscopic level of individual bubbles or parts of bubbles no explicit insight can be gained as to why individual bubbles are in a given global or macro-macroscopic state. Averaging of the macroscopic constitutive equations of each phase leads to a relation between the macroscopic and macro-macroscopic states. However, this procedure is incomplete, because the constitutive equations averaged in this manner lack the complete global information which is necessary for the proper formulation of macro-macroscopic constitutive equations. In a bubbly two-phase flow, the agglomeration or disintegration of bubbles is triggered by the global information produced from the organized character of large number of bubbles or by the conditions specified by the macro-macroscopic state of the flow field.
- 2) Failure to provide a closed system of equations since averaging of the macroscopic constitutive equations leads again to covariance coefficients or turbulent correlation coefficients that need modeling through additional constitutive equations.

The easiest way to construct constitutive equations for multiphase flows is through a phenomenological modeling of individual flow regimes (bubbly flow, annular flow, etc.). This approach has been widely used for the practical design of multiphase flow equipment, and it will continue to be used primarily because of its simplicity and established confidence [9]. For mixtures of three or more phases this approach has not been adequately explored.

In the second approach in the formulation of constitutive equations for multiphase mixtures, where the constitutive equations are deduced from the holistic or macro-macroscopic modeling considerations, two modeling procedures can be also identified. These are: 1) the procedure followed in the classical thermodynamics of irreversible processes [16], and 2) the procedure of single-phase continuum mechanics [13].

As pointed out by Bataille and Kestin [6], the constitutive equations for multiphase mixtures derived on the basis of an approach from the classical thermodynamics of irreversible processes require the existence of a macro-macroscopic fundamental equation of state or of a Gibbs equation. Furthermore, it is also required that the resulting entropy equation exhibits the

bilinear form in terms of fluxes and forces. Marle [8] and Hung [17] utilized this procedure by making a highly questionable assumption on the equality of macroscopic and macro-macroscopic fundamental equations and derived some special form of constitutive equations.

The continuum mechanics approach to study the constitutive equations is refuted by some and praised by others. Those who disapprove this approach find the method "...sufficiently arbitrary and general to account for nearly any kind of situation" [6], or unjustifiable because it is based on questionable principles of material frame invariance, equipresence, positive entropy production and on a number of other points [18]. While it is true that some workers in single-phase continuum mechanics have used the approach beyond physical justification, it still remains to be proved that the general procedure is incorrect. Those who refute the foundational basis of continuum mechanics start from special models of the physical world and argue to prove the trouble spots. While their (reductionistic) argument might be correct, is it not also safe to argue that it is just possible for some information in this approach, perhaps inconsequential at a first glance, to have been neglected which leads to false conclusions? As far as the linear theory of constitutive equations is concerned, there is, however, essentially no opposition to the unsoundness of the continuum mechanics approach, and it is, therefore, in this approximation that certain results for constitutive equations of multiphase flows will be summarized below.

Constitutive equations for multiphase mixtures using the continuum mechanics approach, have been studied to various degrees in References 1, 12, 14, 15 and 19. The important feature in the continuum mechanics approach is that the entropy equation is utilized to obtain restrictions on the constitutive assumption. Based on the field equations extended from the theory of multicomponent mixtures to multiphase mixtures, Drumheller and Bedford [1] and Bowen [12] utilized the entropy equation for the mixture as a whole, while Dobran [14,15] used the single-phase entropy Eq. (27) to restrict the constitutive assumption. Drew and Lahey [19] considered only some general constitutive principles for two-phase flows using the time-averaged model of Ishii [4] and did not use the entropy equation to restrict the form of constitutive assumption. As discussed in Section 2, an entropy equation for each phase is consistent with all averaging approaches while the entropy equation for the mixture as a whole does not appear to be too restrictive and the constitutive equations based on it should then be more general.

In the remaining part of this section, certain results for constitutive equations for a multiphase mixture of fluids from the studies in References 14 and 15 will be presented. For this purpose, it is convenient to eliminate the heat generation rate  $\bar{r}_\alpha$  in the entropy Eq. (27) by using the energy Eq. (26) to obtain

$$-\bar{\rho}_\alpha (\bar{\psi}_\alpha + \bar{\theta}_\alpha \bar{\zeta}_\alpha) + \text{tr}(\bar{\mathbf{T}}_\alpha^T \nabla \bar{\mathbf{v}}_\alpha) - \frac{\bar{\mathbf{q}}_\alpha \cdot \nabla \bar{\theta}_\alpha}{\bar{\theta}_\alpha} + \hat{\epsilon}_\alpha + \bar{\theta}_\alpha \hat{s}_\alpha + \hat{\epsilon}_\alpha \bar{\zeta}_\alpha \bar{\theta}_\alpha \geq 0, \quad (31)$$

where the Helmholtz potential  $\bar{\psi}_\alpha$  is defined as follows:

$$\bar{\psi}_\alpha \equiv \bar{\epsilon}_\alpha - \bar{\theta}_\alpha \bar{\zeta}_\alpha. \quad (32)$$

Equation (31) together with Eqs. (23)-(26) were used by Dobran [14] to study constitutive equations for a dispersed multiphase fluid mixture with a single but nonuniform temperature and without the possibility of phase change, i.e.,

$\bar{\theta}_\alpha = \theta$  and  $\hat{\epsilon}_\alpha = 0$ . The constitutive assumption studied in Reference 14,

$$\left( \bar{\mathbf{T}}_\alpha, \hat{\mathbf{p}}_\alpha, \hat{\mathbf{M}}_\alpha, \bar{\psi}_\alpha, \bar{\mathbf{q}}_\alpha, \hat{\epsilon}_\alpha, \bar{\zeta}_\alpha, \hat{s}_\alpha \right) = f[\theta, \nabla \theta, \bar{\rho}_\beta, \nabla \bar{\rho}_\beta, \mathbf{D}_\beta, \mathbf{W}_\beta - \mathbf{W}_\gamma, \bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma, \bar{\dot{\mathbf{v}}}_\beta - \bar{\dot{\mathbf{v}}}_\gamma - 2 \mathbf{W}_\gamma (\bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma)], \quad (33)$$

also ignores the memory effects but accounts for the nonuniform temperature and densities; for heat conduction,  $\nabla \theta$ ; for density gradients,  $\nabla \bar{\rho}_\beta$ ; for viscous effects,  $\mathbf{F}_\beta$ ; for viscous drag,  $\bar{\mathbf{v}}_\beta$ ; and for virtual mass effects,  $\bar{\dot{\mathbf{v}}}_\beta$ . Note that in the notation of Eq. (33) each constitutive variable of the phase  $\alpha$ ,  $\alpha = 1, \dots, \gamma$ , depends on all the other variables,  $\beta = 1, \dots, \gamma$ , in the functional expression for  $f$ . Using the principle of objectivity or the material frame indifference [7,14,15]: 1)  $\mathbf{F}_\beta$  must be replaced by  $\mathbf{D}_\beta$  and  $\mathbf{W}_\beta - \mathbf{W}_\gamma$  where  $\mathbf{D}_\beta = \frac{1}{2}(\mathbf{L}_\beta + \mathbf{L}_\beta^T)$ ,  $\mathbf{W}_\beta = \frac{1}{2}(\mathbf{L}_\beta - \mathbf{L}_\beta^T)$  and  $\mathbf{L}_\beta = \mathbf{D}_\beta + \mathbf{W}_\beta$  is the velocity gradient,  $\nabla \bar{\mathbf{v}}_\beta$ , 2)  $\bar{\dot{\mathbf{v}}}_\beta$  must be replaced by  $\bar{\dot{\mathbf{v}}}_\beta - \bar{\dot{\mathbf{v}}}_\gamma$ , and 3)  $\bar{\dot{\mathbf{v}}}_\beta$  must appear in the frame invariant form  $\bar{\dot{\mathbf{v}}}_\beta - \bar{\dot{\mathbf{v}}}_\gamma - 2 \mathbf{W}_\gamma (\bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma)$ . By substituting Eq. (33) into Eq. (31) the following is established:

1. The equilibrium state of the multiphase mixture specified by the conditions  $\nabla \theta = \mathbf{0}$ ,  $\mathbf{D}_\beta = \mathbf{0}$ ,  $\mathbf{W}_\beta = \mathbf{0}$ ,  $\bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma = \mathbf{0}$ ,  $\bar{\dot{\mathbf{v}}}_\beta = \mathbf{0}$ , for  $\beta = 1, \dots, \gamma$ .
2. The restrictions on the equilibrium state coefficients in the linearized constitutive equations.

The above procedure is long and it leads to the following linearized constitutive equations:

$$\bar{\psi}_\alpha = \bar{\psi}_\alpha(\theta, \bar{\rho}_\alpha), \quad \bar{\zeta}_\alpha = -\frac{\partial \bar{\psi}_\alpha(\theta, \bar{\rho}_\alpha)}{\partial \theta} \quad (34)$$

$$\hat{s}_\alpha = 0, \quad \hat{\epsilon}_\alpha = 0 \quad (35)$$

$$\bar{\pi}_\alpha \mathbf{I} \equiv -\bar{\mathbf{T}}_\alpha[\theta, \nabla \theta = \mathbf{0}, \bar{\rho}_\beta, \nabla \bar{\rho}_\beta = \mathbf{0}, \mathbf{D}_\beta = \mathbf{0}, \mathbf{W}_\beta - \mathbf{W}_\gamma = \mathbf{0}, \bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma = \mathbf{0}, \bar{\dot{\mathbf{v}}}_\beta - \bar{\dot{\mathbf{v}}}_\gamma - 2 \mathbf{W}_\gamma (\bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma) = \mathbf{0}] \quad (36)$$

$$\hat{\mathbf{p}}_\alpha = -\gamma_\alpha \nabla \theta - \sum_{\beta=1}^{\gamma-1} \epsilon_{\alpha\beta} (\bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma) - \sum_{\beta=1}^{\gamma-1} \Delta_{\alpha\beta} (\bar{\dot{\mathbf{v}}}_\beta - \bar{\dot{\mathbf{v}}}_\gamma) \quad (37)$$

$$\hat{\mathbf{M}}_\alpha = \bar{\mathbf{T}}_\alpha - \bar{\mathbf{T}}_\alpha^T = 4 \sum_{\beta=1}^{\gamma-1} \phi_{\alpha\beta} (\mathbf{W}_\beta - \mathbf{W}_\gamma) \quad (38)$$

$$\bar{\mathbf{T}}_\alpha = -\bar{\pi}_\alpha \mathbf{I} + \sum_{\beta=1}^{\gamma} [\lambda_{\alpha\beta} (\text{tr } \mathbf{D}_\beta) \mathbf{I} + 2\mu_{\alpha\beta} \mathbf{D}_\beta] + \frac{2\sum_{\beta=1}^{\gamma-1} \phi_{\alpha\beta} (\mathbf{W}_\beta - \mathbf{W}_\gamma)}{\beta=1} \quad (39)$$

$$\bar{\mathbf{q}}_\alpha = -\kappa_\alpha \nabla \theta - \sum_{\beta=1}^{\gamma-1} \zeta_{\alpha\beta} (\bar{\mathbf{v}}_\beta - \bar{\mathbf{v}}_\gamma) - \sum_{\beta=1}^{\gamma-1} \nu_{\alpha\beta} (\bar{\dot{\mathbf{v}}}_\beta - \bar{\dot{\mathbf{v}}}_\gamma) - \bar{\rho}_\alpha \bar{\zeta}_\alpha \theta \mathbf{u}_\alpha, \quad (40)$$

where the coefficients  $\gamma_\alpha$ ,  $\epsilon_{\alpha\beta}$ ,  $\Delta_{\alpha\beta}$ ,  $\phi_{\alpha\beta}$ ,  $\lambda_{\alpha\beta}$ ,  $\mu_{\alpha\beta}$ ,  $\kappa_\alpha$ ,  $\zeta_{\alpha\beta}$  and  $\nu_{\alpha\beta}$  depend on  $\theta$ ,  $\bar{\rho}_1, \dots, \bar{\rho}_\gamma$ . These coefficients are restricted by the entropy Eq. (31), and for a two-phase fluid mixture, it is shown by Dobran [14,15] that the

results are in accord with the available experimental data.

For a two-phase highly dispersed fluid mixture where the linear momentum supply for the mixture  $\hat{\mathbf{p}}_m = \mathbf{0}$ , the angular momentum supply for the mixture  $\hat{\mathbf{M}} = \mathbf{0}$ , the energy supply for the mixture  $\hat{\mathbf{e}}_m = 0$  and the entropy supply for the mixture  $\hat{\mathbf{s}}_m = 0$ , it is shown by Dobran [15] that

- 1)  $\hat{\mathbf{M}}_\alpha = \mathbf{0}$  and thus from Eq. (38) we have the result that the stress tensor for phase  $\alpha$  is symmetric, i.e.,

$$\bar{\mathbf{T}}_\alpha = \bar{\mathbf{T}}_\alpha^T, \quad \phi_{\alpha 1} = 0 \quad (41)$$

- 2) Using Eq. (41) in Eq. (39) the stress tensor for phase  $\alpha$  thus becomes:

$$\bar{\mathbf{T}}_\alpha = -\bar{\pi}_\alpha \mathbf{I} + \lambda_{\alpha\alpha} (\text{tr} \mathbf{D}_\alpha) \mathbf{I} + 2\mu_{\alpha\alpha} \mathbf{D}_\alpha \quad (42)$$

The stress tensor for phase  $\alpha$ ,  $\bar{\mathbf{T}}_\alpha$ , is thus shown to be symmetric only in the special case of disperse flow and it does not depend on the variables from the other phase, ( $\mathbf{W}_1 - \mathbf{W}_2$ ). In their studies of two-phase flow constitutive equations, Ishii [4] and Bataille and Kestin [6] assumed a priori the symmetry of  $\bar{\mathbf{T}}_\alpha$ , while Drew [20] invoked the principle of phase separation in which the bulk phase variables (such as  $\bar{\mathbf{T}}_\alpha$  and  $\bar{\mathbf{q}}_\alpha$ ) are allowed to depend only on the variables from the same phase. Although Drew's principle of phase separation might be correct in special circumstances (cit. the result above expressed by Eq. (42)), within the present theory, however, this principle remains questionable since it is in conflict with Eqs. (38) - (40). In the time averaging approach [4]: 1) the coefficient  $\lambda_{\alpha\alpha} = 0$  in Eq. (42); 2) no account is taken in the theory for virtual mass effects; and 3) the Dufour effect in the expression for the heat flux vector is associated with the concentration gradient and not with the diffusion of phases as represented by Eq. (40) and also agreed upon in the theory of single-phase multicomponent mixtures [10].

The Helmholtz potential in Eq. (34),  $\bar{\psi}_\alpha = \bar{\psi}_\alpha(\Theta, \bar{\rho}_\alpha)$ , represents a fundamental relation of the macro-macroscopic state with the assumptions which lead to the constitutive Eqs. (34) - (40). The knowledge of such a relationship and the coefficients  $\gamma_\alpha$ ,  $\xi_{\alpha\beta}$ ,  $\Delta_{\alpha\beta}$ ,  $\phi_{\alpha\beta}$ ,  $\lambda_{\alpha\beta}$ ,  $\mu_{\alpha\beta}$ ,  $\kappa_\alpha$ ,  $\tau_{\alpha\beta}$  and  $\nu_{\alpha\beta}$  for  $\alpha, \beta = 1, \dots, \gamma$  as functions of  $\Theta, \rho_1, \dots, \rho_\gamma$  are sufficient to close the multiphase field Eqs. (23) - (27) when the constitutive Eqs. (37) - (40) are utilized. Unfortunately, this macro-macroscopic fundamental equation is not known at the present time and this is where the holistic or continuum mechanics approach presented in the paper becomes inadequate. To get around this problem, the constitutive assumption can be enlarged and additional transport equations can be assumed [12,21].

From the above, it is clear that the linearized theory of constitutive equations yields many results that are in accord with past modeling practices in two-phase flows, and it also gives more general results than are predicted by these models. The fundamental problem thus remains: should there be a meaningful physical reconciliation between the macroscopic and macro-macroscopic states or should there exist additional transport equations. This problem is, at the present time, the main stumbling block towards the construction of a viable theory for multiphase flows.

#### 4. SUMMARY AND CONCLUSIONS

The theory of multiphase flows is in an infancy of development. While a

general accord exists on the form of conservation and balance equations for multiphase flows, different averaging procedures lead to the definition of different multiphase flow variables which are still controversial. Averaging over time and space domains gives rise to different models and especially manifests itself in the turbulent field equations. The volume-averaged multiphase field equations appear to be the most desirable to motivate the construction of a physically meaningful and mathematically rigorous set of conservation and balance equations. The field equations based on this approach were presented in the paper where it is also shown that the averaging or reductionistic approach utilized to construct conservation, balance and constitutive equations for multiphase mixtures needs reconciliations between two different levels of description: the macroscopic and macro-macroscopic levels. To avoid this, a holistic or continuum mechanics approach is more desirable since it yields a more consistent theory.

The results for constitutive equations, using the continuum mechanics approach and discussed in the paper, should, through fundamental experiments, lead to better modeling practices. Other procedures such as flow regime modeling will continue to provide special forms of the constitutive equations and, therefore, the verification of various theoretical models. The approach, based on irreversible thermodynamics, lacks the appropriate macro-macroscopic fundamental equation and, for this reason, it has not to date found a wide support. Clearly, while considerable progress has been made in the formulation of conservation, balance and constitutive equations for multiphase flows, much more work remains to be done in the future through both analysis and fundamental experiments.

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#### APPENDIX A

The conservation and balance equations for a multiphase mixture are obtained by summing up each of the Eqs. (23) - (27) and by utilizing Eq. (22), i.e.

$$\text{Mass} \quad \dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0 \quad (\text{A.1})$$

$$\text{Linear Momentum:} \quad \rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{b} + \hat{\mathbf{p}}_m \quad (\text{A.2})$$

$$\text{Angular Momentum:} \quad \hat{\mathbf{M}}_m = \mathbf{T} - \mathbf{T}^T \quad (\text{A.3})$$

$$\text{Energy:} \quad \rho \dot{\epsilon} = \text{tr}(\mathbf{T}^T \nabla \mathbf{v}) - \nabla \cdot \mathbf{q} + \rho r + (\hat{\epsilon}_m - \mathbf{v} \cdot \hat{\mathbf{p}}_m) \quad (\text{A.4})$$

$$\text{Entropy:} \quad \rho \dot{s} + \nabla \cdot \sum_{\alpha=1}^Y \frac{\mathbf{q}_\alpha}{\bar{\theta}_\alpha} + \bar{\rho}_\alpha \tilde{s}_\alpha \mathbf{u}_\alpha - \sum_{\alpha=1}^Y \frac{\bar{\rho}_\alpha \tilde{r}_\alpha}{\bar{\theta}_\alpha} + \hat{s}_m \geq 0 \quad (\text{A.5})$$

where the compatibility conditions between the phasic Eqs. (23) - (27) and mixture Eqs. (A.1) - (A.5) are expressed by the following relations:

$$\sum_{\alpha=1}^Y \hat{\epsilon}_\alpha = 0 \quad (\text{A.6})$$

$$\mathbf{T} \equiv \sum_{\alpha=1}^Y (\bar{\mathbf{T}}_\alpha - \bar{\rho}_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) \quad (\text{A.7})$$

$$\rho \mathbf{b} \equiv \sum_{\alpha=1}^Y \bar{\rho}_\alpha \tilde{\mathbf{b}}_\alpha \quad (\text{A.8})$$

$$\hat{\mathbf{p}}_m \equiv \sum_{\alpha=1}^Y (\hat{\epsilon}_\alpha \mathbf{u}_\alpha + \hat{\mathbf{p}}_\alpha) \quad (\text{A.9})$$

$$\hat{\mathbf{M}}_m \equiv \sum_{\alpha=1}^Y \hat{\mathbf{M}}_\alpha \quad (\text{A.10})$$

$$\rho \epsilon \equiv \sum_{\alpha=1}^Y (\bar{\rho}_\alpha \tilde{\epsilon}_\alpha + \frac{1}{2} \bar{\rho}_\alpha \mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \quad (\text{A.11})$$

$$\rho r \equiv \sum_{\alpha=1}^Y (\bar{\rho}_\alpha \tilde{r}_\alpha + \bar{\rho}_\alpha \tilde{\mathbf{b}}_\alpha \cdot \mathbf{u}_\alpha) \quad (\text{A.12})$$

$$\mathbf{q} \equiv \sum_{\alpha=1}^Y [\bar{\mathbf{q}}_\alpha - \bar{\mathbf{T}}_\alpha^T \mathbf{u}_\alpha + \bar{\rho}_\alpha \tilde{\epsilon}_\alpha \mathbf{u}_\alpha + \frac{1}{2} \bar{\rho}_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \mathbf{u}_\alpha] \quad (\text{A.13})$$

$$\hat{\epsilon}_m - \mathbf{v} \cdot \hat{\mathbf{p}}_m \equiv \sum_{\alpha=1}^Y [\hat{\epsilon}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{p}}_\alpha + \hat{\epsilon}_\alpha (\tilde{\epsilon}_\alpha + \frac{1}{2} \mathbf{u}_\alpha \cdot \mathbf{u}_\alpha)] \quad (\text{A.14})$$

$$\rho s \equiv \sum_{\alpha=1}^Y \bar{\rho}_\alpha \tilde{s}_\alpha \quad (\text{A.15})$$

$$\hat{s}_m \equiv \sum_{\alpha=1}^Y \hat{s}_\alpha \quad (\text{A.16})$$

The mixture source terms  $\hat{\mathbf{p}}_m$ ,  $\hat{\mathbf{M}}_m$ ,  $\hat{\epsilon}_m$  and  $\hat{s}_m$  account for two effects [7]: 1) for the interfacial forces between the phases in the averaging volume  $V$ ; and 2) for the covariance effect or the nonlocal material response. Notice also that Eqs. (23) - (27) and (A.1) - (A.5) can be time-averaged to obtain turbulent field equations.